Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,7.1.4 7.2.1 7.3.3 7.4.4 7.5.2 7.6.4 7.6.5 7.6.7 7.6.8 7.7.2 and 7.8.1

Problem 7.1.4

Theorem 7.2 which says that

$$\operatorname{Var}[W] = \operatorname{Var}[X] + \operatorname{Var}[Y] + 2\operatorname{Cov}[X,Y]$$

The first two moments of X are

$$E[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3$$
$$E\left[X^2\right] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6$$

Thus the variance of X is $\operatorname{Var}[X] = E\left[X^2\right] - (E[X])^2 = 1/18$. By symmetry, it should be apparent that $E[Y] = E[X] = 1/3$ and $\operatorname{Var}[Y] = \operatorname{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = 1/12$$

The covariance is

$$\operatorname{Cov}[X,Y] = E[XY] - E[X]E[Y] = 1/12 - (1/3)^2 = -1/36$$

Finally, the variance of the sum $W = X + Y$ is

$$\operatorname{Var}[W] = \operatorname{Var}[X] + \operatorname{Var}[Y] - 2\operatorname{Cov}[X,Y] = 2/18 - 2/36 = 1/18$$

For this specific problem, it's arguable whether it would easier to find $\operatorname{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \le w \le 1$,

$$F_W(w) = P[X + Y \le w] = \int_0^w \int_0^{w-x} 2\,dy\,dx = \int_0^w 2(w-x)\,dx = w^2$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases}$$

From the PDF, the first and second moments of W are

$$E[W] = \int_0^1 2w^2\,dw = 2/3 \qquad E\left[W^2\right] = \int_0^1 2w^3\,dw = 1/2$$

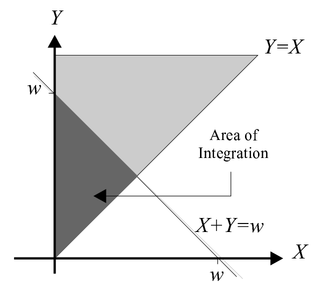
The variance of W is $\operatorname{Var}[W] = E\left[W^2\right] - (E[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 7.2.1

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

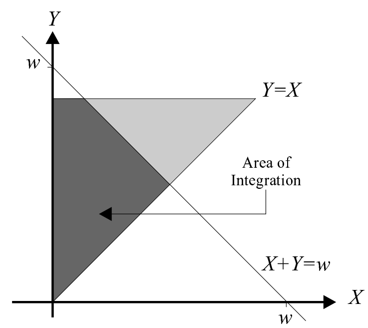
We wish to find the PDF of W where $W = X + Y$. First we find the CDF of W, $F_W(w)$, but we must realize that the CDF will require different integrations for different values of w.

For values of $0 \le w \le 1$ we look to integrate the shaded area in the figure to the right.



$$F_W(w) = \int_0^{\frac{w}{2}} \int_x^{w-x} 2\,dy\,dx = \frac{w^2}{2}$$

For values of w in the region $1 \le w \le 2$ we look to integrate over the shaded region in the graph to the right. From the graph we see that we can integrate with respect to x first, ranging y from 0 to $w/2$, thereby covering the lower right triangle of the shaded region and leaving the upper trapezoid, which is accounted for in the second term of the following expression:



$$\begin{aligned} F_W(w) &= \int_0^{\frac{w}{2}} \int_0^{y} 2\,dx\,dy + \int_{\frac{w}{2}}^{1} \int_0^{w-y} 2\,dx\,dy \\ &= 2w - 1 - \frac{w^2}{2} \end{aligned}$$

Putting all the parts together gives:

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w^2}{2} & 0 \le w \le 1 \\ 2w - 1 - \frac{w^2}{2} & 1 \le w \le 2 \\ 1 & w > 2 \end{cases}$$

And the PDF is found by taking the derivative with respect to w:

$$f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 7.3.3

$$P_K(k) = \begin{cases} 1/n & k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

The corresponding MGF of K is

$$\begin{aligned}\phi_K(s) = E\left[e^{sK}\right] &= \frac{1}{n}\left(e^s + e^2 s + \cdots + e^{ns}\right) \\ &= \frac{e^s}{n}\left(1 + e^s + e^{2s} + \cdots + e^{(n-1)s}\right) \\ &= \frac{e^s(e^{ns}-1)}{n(e^s-1)}\end{aligned}$$

We can evaluate the moments of K by taking derivatives of the MGF. Some algebra will show that

$$\frac{d\phi_K(s)}{ds} = \frac{ne^{(n+2)s} - (n+1)e^{(n+1)s} + e^s}{n(e^s-1)^2}$$

Evaluating $d\phi_K(s)/ds$ at $s = 0$ yields $0/0$. Hence, we apply l'Hôpital's rule twice (by twice differentiating the numerator and twice differentiating the denominator) when we write

$$\begin{aligned}\left.\frac{d\phi_K(s)}{ds}\right|_{s=0} &= \lim_{s\to 0} \frac{n(n+2)e^{(n+2)s} - (n+1)^2 e^{(n+1)s} + e^s}{2n(e^s-1)} \\ &= \lim_{s\to 0} \frac{n(n+2)^2 e^{(n+2)s} - (n+1)^3 e^{(n+1)s} + e^s}{2ne^s} \\ &= (n+1)/2\end{aligned}$$

A significant amount of algebra will show that the second derivative of the MGF is

$$\frac{d^2\phi_K(s)}{ds^2} = \frac{n^2 e^{(n+3)s} - (2n^2+2n-1)e^{(n+2)s} + (n+1)^2 e^{(n+1)s} - e^{2s} - e^s}{n(e^s-1)^3}$$

Evaluating $d^2\phi_K(s)/ds^2$ at $s = 0$ yields $0/0$. Because $(e^s-1)^3$ appears in the denominator, we need to use l'Hôpital's rule three times to obtain our answer.

$$\begin{aligned}\left.\frac{d^2\phi_K(s)}{ds^2}\right|_{s=0} &= \lim_{s\to 0} \frac{n^2(n+3)^3 e^{(n+3)s} - (2n^2+2n-1)(n+2)^3 e^{(n+2)s} + (n+1)^5 - 8e^{2s} - e^s}{6ne^s} \\ &= \frac{n^2(n+3)^3 - (2n^2+2n-1)(n+2)^3 + (n+1)^5 - 9}{6n} \\ &= (2n+1)(n+1)/6\end{aligned}$$

We can use these results to derive two well known results. We observe that we can directly use the PMF $P_K(k)$ to calculate the moments

$$E[K] = \frac{1}{n}\sum_{k=1}^{n} k \qquad E\left[K^2\right] = \frac{1}{n}\sum_{k=1}^{n} k^2$$

Using the answers we found for $E[K]$ and $E\left[K^2\right]$, we have the formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 7.4.4

By Theorem 7.10, we know that $\phi_M(s) = [\phi_K(s)]^n$.

(a) The first derivative of $\phi_M(s)$ is

$$\frac{d\phi_M(s)}{ds} = n[\phi_K(s)]^{n-1}\frac{d\phi_K(s)}{ds}$$

We can evaluate $d\phi_M(s)/ds$ at $s = 0$ to find $E[M]$.

$$E[M] = \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} = n[\phi_K(s)]^{n-1}\left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = nE[K]$$

(b) The second derivative of $\phi_M(s)$ is

$$\frac{d^2\phi_M(s)}{ds^2} = n(n-1)[\phi_K(s)]^{n-2}\left(\frac{d\phi_K(s)}{ds}\right)^2 + n[\phi_K(s)]^{n-1}\frac{d^2\phi_K(s)}{ds^2}$$

Evaluating the second derivative at $s = 0$ yields

$$E\left[M^2\right] = \left.\frac{d^2\phi_M(s)}{ds^2}\right|_{s=0} = n(n-1)\left(E[K]\right)^2 + nE\left[K^2\right]$$

Problem 7.5.2

Using the moment generating function of X, $\phi_X(s) = e^{\sigma^2 s^2/2}$. We can find the nth moment of X, $E[X^n]$ by taking the nth derivative of $\phi_X(s)$ and setting $s = 0$.

$$\begin{aligned}
E[X] &= \left.\sigma^2 s e^{\sigma^2 s^2/2}\right|_{s=0} = 0 \\
E\left[X^2\right] &= \left.\sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2}\right|_{s=0} = \sigma^2
\end{aligned}$$

Continuing in this manner we find that

$$\begin{aligned}
E\left[X^3\right] &= \left.\left(3\sigma^4 s + \sigma^6 s^3\right) e^{\sigma^2 s^2/2}\right|_{s=0} = 0 \\
E\left[X^4\right] &= \left.\left(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4\right) e^{\sigma^2 s^2/2}\right|_{s=0} = 3\sigma^4
\end{aligned}$$

Problem 7.6.4

random sum of random variables

$$V + Y_1 + \cdots + Y_K$$

where Y_i has the exponential PDF

$$f_{Y_i}(y) = \begin{cases} \frac{1}{15}e^{-y/15} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Following Examples 7.7 and 7.10, the MGFs of Y and K are

$$\phi_Y(s) = \frac{1/15}{1/15 - s} = \frac{1}{1-15s} \qquad \phi_K(s) = e^{20(e^s-1)}$$

From Theorem 7.14, V has MGF

$$\phi_V(s) = \phi_K(\ln \phi_Y(s)) = e^{20(\phi_Y(s)-s)} = e^{300s/(1-15s)}$$

The PDF of V cannot be found in a simple form. However, we can use the MGF to calculate the mean and variance. In particular,

$$\begin{aligned}
E[V] &= \left.\frac{d\phi_V(s)}{ds}\right|_{s=0} = \left. e^{300s/(1-15s)} \frac{300}{(1-15s)^2}\right|_{s=0} = 300 \\
E\left[V^2\right] &= \left.\frac{d^2\phi_V(s)}{ds^2}\right|_{s=0} \\
&= \left. e^{300s/(1-15s)} \left(\frac{300}{(1-15s)^2}\right)^2 + e^{300s/(1-15s)} \frac{9000}{(1-15s)^3}\right|_{s=0} = 99,000
\end{aligned}$$

Thus, V has variance $\text{Var}[V] = E\left[V^2\right] - (E[V])^2 = 9,000$ and standard deviation $\sigma_V \approx 94.9$.

A second way to calculate the mean and variance of V is to use Theorem 7.15 which says

$$\begin{aligned}
E[V] &= E[K]E[Y] = 20(15) = 200 \\
\text{Var}[V] &= E[K]\,\text{Var}[Y] + \text{Var}[K](E[Y])^2 = (20)15^2 + (20)15^2 = 9000
\end{aligned}$$

Problem 7.6.5

have one of $\binom{46}{6}$ combinations, the probability a ticket is a winner is

$$q = \frac{1}{\binom{46}{6}}$$

Let $X_i = 1$ if the ith ticket sold is a winner; otherwise $X_i = 0$. Since the number K of tickets sold has a Poisson PMF with $E[K] = r$, the number of winning tickets is the random sum

$$V = X_1 + \cdots + X_K$$

From Appendix A,

$$\phi_X(s) = (1-q) + qe^s \qquad \phi_K(s) = e^{r[e^s-1]}$$

By Theorem 7.14,

$$\phi_V(s) = \phi_K(\ln \phi_X(s)) = e^{r[\phi_X(s)-1]} = e^{rq(e^s-1)}$$

Hence, we see that V has the MGF of a Poisson random variable with mean $E[V] = rq$. The PMF of V is

$$P_V(v) = \begin{cases} (rq)^v e^{-rq}/v! & v = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Problem 7.6.7

The way to solve for the mean and variance of U is to use conditional expectations. Given $K = k$, $U = X_1 + \cdots + X_k$ and

$$E[U|K=k] = E[X_1 + \cdots + X_k | X_1 + \cdots + X_n = k] = \sum_{i=1}^{k} E[X_i | X_1 + \cdots + X_n = k]$$

Since X_i is a Bernoulli random variable,

$$E[X_i | X_1 + \cdots + X_n = k] = P\left[X_i = 1 | \sum_{j=1}^{n} X_j = k\right] = \frac{P\left[X_i = 1, \sum_{j \neq i} X_j = k-1\right]}{P\left[\sum_{j=1}^{n} X_j = k\right]}$$

Note that $\sum_{j=1}^{n} X_j$ is just a binomial random variable for n trials while $\sum_{j \neq i} X_j$ is a binomial random variable for $n-1$ trials. In addition, X_i and $\sum_{j \neq i} X_j$ are independent random variables. This implies

$$\begin{aligned}
E[X_i | X_1 + \cdots + X_n = k] &= \frac{P[X_i = 1] P\left[\sum_{j \neq i} X_j = k-1\right]}{P\left[\sum_{j=1}^{n} X_j = k\right]} \\
&= \frac{p \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)}}{\binom{n}{k} p^k (1-p)^{n-k}} \\
&= \frac{k}{n}
\end{aligned}$$

A second way to find this result is to use symmetry to argue that $E[X_i | X_1 + \cdots + X_n = k]$ should be the same for each i. In particular, if we say $E[X_i | X_1 + \cdots + X_n = k] = \gamma$, then

$$n\gamma = \sum_{i=1}^{n} E[X_i | X_1 + \cdots + X_n = k] = E[X_1 + \cdots + X_n | X_1 + \cdots + X_n = k] = k$$

Thus $\gamma = k/n$. At any rate, the conditional mean of U is

$$E[U|K=k] = \sum_{i=1}^{k} E[X_i | X_1 + \cdots + X_n = k] = \sum_{i=1}^{k} \frac{k}{n} = \frac{k^2}{n}$$

This says that the random variable $E[U|K] = K^2/n$. Using iterated expectations, we have

$$E[U] = E[E[U|K]] = E\left[K^2/n\right]$$

Since K is a binomial random varaiable, we know that $E[K] = np$ and $\text{Var}[K] = np(1-p)$. Thus,

$$E[U] = \frac{1}{n} E\left[K^2\right] = \frac{1}{n}\left(\text{Var}[K] + (E[K])^2\right) = p(1-p) + np^2$$

On the other hand, V is just and ordinary random sum of independent random variables and the mean of $E[V] = E[X]E[M] = np^2$.

Problem 7.6.8

played, we can write the total number of points earned as the random sum

$$Y = X_1 + X_2 + \cdots + X_N$$

(a) It is tempting to use Theorem 7.14 to find $\phi_Y(s)$; however, this would be wrong since each X_i is not independent of N. In this problem, we must start from first principles using iterated expectations.

$$\phi_Y(s) = E\left[E\left[e^{s(X_1+\cdots+X_N)}|N\right]\right] = \sum_{n=1}^{\infty} P_N(n)\, E\left[e^{s(X_1+\cdots+X_n)}|N=n\right]$$

Given $N = n$, $X_1, \ldots, X_n$ are independent so that

$$E\left[e^{s(X_1+\cdots+X_n)}|N=n\right] = E\left[e^{sX_1}|N=n\right]E\left[e^{sX_2}|N=n\right]\cdots E\left[e^{sX_n}|N=n\right]$$

Given $N = n$, we know that games 1 through $n-1$ were either wins or ties and that game n was a loss. That is, given $N = n$, $X_n = 0$ and for $i < n$, $X_i \neq 0$. Moreover, for $i < n$, X_i has the conditional PMF

$$P_{X_i|N=n}(x) = P_{X_i|X_i\neq 0}(x) = \begin{cases} 1/2 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

These facts imply

$$E\left[e^{sX_n}|N=n\right] = e^0 = 1$$

and that for $i < n$,

$$E\left[e^{sX_i}|N=n\right] = (1/2)e^s + (1/2)e^{2s} = e^s/2 + e^{2s}/2$$

Now we can find the MGF of Y.

$$\begin{aligned}
\phi_Y(s) &= \sum_{n=1}^{\infty} P_N(n)\, E\left[e^{sX_1}|N=n\right]E\left[e^{sX_2}|N=n\right]\cdots E\left[e^{sX_{n-1}}\right]E\left[e^{sX_n}|N=n\right] \\
&= \sum_{n=1}^{\infty} P_N(n) \left[e^s/2 + e^{2s}/2\right]^{n-1} \\
&= \frac{1}{e^s/2 + e^{2s}/2} \sum_{n=1}^{\infty} P_N(n) \left[e^s/2 + e^{2s}/2\right]^{n} \\
&= \frac{1}{e^s/2 + e^{2s}/2} \sum_{n=1}^{\infty} P_N(n)\, e^{n \ln[(e^s+e^{2s})/2]} \\
&= \frac{\phi_N(\ln[e^s/2 + e^{2s}/2])}{e^s/2 + e^{2s}/2}
\end{aligned}$$

The tournament ends as soon as you lose a game. Since each game is a loss with probability $1/3$ independent of any previous game, the number of games played has the geometric PMF and corresponding MGF

$$P_N(n) = \begin{cases} (2/3)^{n-1}(1/3) & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N(s) = \frac{(1/3)e^s}{1-(2/3)e^s}$$

Thus, the MGF of Y is

$$\phi_Y(s) = \frac{1/3}{1 - (e^s + e^{2s})/3}$$

(b) To find the moments of Y, we evaluate the derivatives of the MGF $\phi_Y(s)$. Since

$$\frac{d\phi_Y(s)}{ds} = \frac{e^s + 2e^{2s}}{9\left[1 - e^s/3 - e^{2s}/3\right]^2}$$

we see that

$$E[Y] = \left.\frac{d\phi_Y(s)}{ds}\right|_{s=0} = \frac{3}{9(1/3)^2} = 3$$

If you're curious, you may notice that $E[Y] = 3$ precisely equals $E[N]E[X_i]$, the answer you would get if you mistakenly assumed that N and each X_i were independent. Although this may seem like a coincidence, its actually the result of theorem known as Wald's equality.

The second derivative of the MGF is

$$\frac{d^2\phi_Y(s)}{ds^2} = \frac{(1 - e^s/3 - e^{2s}/3)(e^s + 4e^{2s}) + 2(e^s + 2e^{2s})^2/3}{9(1 - e^s/3 - e^{2s}/3)^3}$$

The second moment of Y is

$$E\left[Y^2\right] = \left.\frac{d^2\phi_Y(s)}{ds^2}\right|_{s=0} = \frac{5/3 + 6}{1/3} = 23$$

The variance of Y is $\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 23 - 9 = 14$.

Problem 7.7.2

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable D_i as the number of data calls in a single telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_i's are independent and identically distributed withe the following PMF.

$$P_D(d) = \begin{cases} 0.8 & d = 0 \\ 0.2 & d = 1 \\ 0 & \text{otherwise} \end{cases}$$

From the above we can determine that

$$E[D] = 0.2 \qquad \text{Var}[D] = 0.2 - 0.04 = 0.16$$

With the previous descriptions, we can answer the following questions.

(a) $E[K_{100}] = 100E[D] = 20$

(b) $\text{Var}[K_{100}] = \sqrt{100\,\text{Var}[D]} = \sqrt{16} = 4$

(c) $P[K_{100} \geq 18] = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-1/2) = \Phi(1/2) = 0.6915$

(d) $P[16 \leq K_{100} \leq 24] = \Phi\left(\frac{24-20}{4}\right) - \Phi\left(\frac{16-20}{4}\right) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826$

Problem 7.8.1

In Example 7.12, we learned that a sum of iid Poisson random variables is a Poisson random variable. Hence W_n is a Poisson random variable with mean $E[W_n] = nE[K] = n$. Thus W_n has variance $\text{Var}[W_n] = n$ and PMF

$$P_{W_n}(w) = \begin{cases} n^w e^{-n}/w! & w = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

All of this implies that we can exactly calculate

$$P[W_n = n] = P_{W_n}(n) = n^n e^{-n}/n!$$

Since we can perform the exact calculation, using a central limit theorem may seem silly; however for large n, calculating n^n or $n!$ is difficult for large n. Moreover, it's interesting to see how good the approximation is. In this case, the approximation is

$$P[W_n = n] = P[n \le W_n \le n] \approx \Phi\left(\frac{n + 0.5 - n}{\sqrt{n}}\right) - \Phi\left(\frac{n - 0.5 - n}{\sqrt{n}}\right) = 2\Phi\left(\frac{1}{2\sqrt{n}}\right) - 1$$

The comparison of the exact calculation and the approximation are given in the following table.

$P[W_n = n]$	$n = 1$	$n = 4$	$n = 16$	$n = 64$
exact	0.3679	0.1954	0.0992	0.0498
approximate	0.3829	0.1974	0.0995	0.0498