1. **(50 points) Amplitude Modulation**

(a) **(20 points)** Let \( m(t) = e^{-t}u(t) \) where \( u(t) \) is the unit step function. Let \( r(t) = m(t)\cos(2\pi f_ct) \) where \( f_c \gg 1 \). Provide an analytic expression for \( R(f) \) the Fourier transform of \( r(t) \) and carefully sketch \( |R(f)| \).

**SOLUTION:**

\[
M(f) = \frac{1}{j2\pi f + 1}
\]

\[
R(f) = \frac{1}{2}M(f-f_c)+\frac{1}{2}M(f+f_c) = \frac{1}{2}\left[ \frac{1}{j2\pi(f-f_c)+1} + \frac{1}{j2\pi(f+f_c)+1} \right]
\]

\[
|R(f)|^2 = \frac{1}{2}\left[ \frac{1}{(2\pi(f-f_c))^2+1} + \frac{1}{(2\pi(f+f_c))^2+1} \right]
\]

Sketch: two humps, centered at \( f_c \) and \(-f_c\).

(b) **(10 points)** Carefully describe at least two different methods by which \( m(t) \) can be recovered from \( r(t) \). You must justify your answers.

**SOLUTION:** Bandwidth of signal is low compared to \( f_c \), so synchronous AM using \( \cos(2\pi f_c t) \) (multiplication followed by a low pass filter). Or since \( m(t) > 0 \) use an envelope detector.

(c) **(10 points)** Let

\[
g(t) = \sqrt{\frac{1}{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}
\]

with Fourier transform

\[
G(f) = e^{-f^2\sigma^2}
\]

We form a periodic carrier signal

\[
\phi(t) = \sum_{k=-\infty}^{\infty} g(t - \frac{k}{f_c})
\]

where \( f_c \) is the (fundamental) carrier frequency. Assume some general (i.e., different from the previous parts) program material \( m(t) \) band limited to \( B \ll f_c \). If \( \sigma \ll \frac{1}{f_c} \), can \( m(t) \) always be recovered from \( r(t) = m(t)\phi(t) \) using an envelope detector? Carefully and analytically discuss why/why not.
SOLUTION: \( \phi(t) \) is a periodic sequence of pulses, so the Fourier transform of \( r(t) \) will be a set of scaled replicas of \( M(f) \). So, we could just low pass filter and get \( M(f) \) back alone. However, we could also look at this another way. We’ve multiplied \( m(t) \) by a set of non-negative “spikes” (the Gaussian function \( g(t) \) looks more and more like an impulse, the smaller \( \sigma \) gets). So “connecting the dots” of \( m(t) \phi(t) \) directly will give us back \( m(t) \). HOWEVER, if \( m(t) \) can be negative and we RECTIFY \( r(t) \), we end up with \( |m(t)\phi(t)| = |m(t)|\phi(t) \) and the envelope detector will recover \( |m(t)| \) not \( m(t) \).

(d) (10 points) Suppose the diode in the envelope detector is faulty and functions as a simple resistor. Is the output of the envelope detector still \( m(t) \)? Always or never?

SOLUTION: Now we don’t rectify. The filter part of the envelope detector will “connect the dots” or “ski the peaks” and we’ll obtain \( m(t) \) directly – always.

2. (50 points) AM/FM/SchmeshM:

Hermaphrodites Inc. has just introduced their new modulation system. They take band limited non-negative program material \( m(t) \) and form the signal

\[
r(t) = m(t) \sin \left( 2\pi f_c t + \beta \int_0^t m(\tau)d\tau \right)
\]

We assume that the bandwidth of \( m(t) \) is \( B \ll f_c \) and \( \beta \) is arbitrary so long as \( \sin \left( 2\pi f_c t + \beta \int_0^t m(\tau)d\tau \right) \) is always a high speed signal.

(a) (25 points) Is \( r(t) \) an example of amplitude modulation? What is the output of an envelope detector with \( r(t) \) as its input? Justify your answer.

SOLUTION: As the title of the problem suggests, this could be both. The multiplier outside the sinusoid definitely modulates the high speed signal \( \sin() \). However, the program material also appears in the argument of \( \sin() \). Now as for the output of an envelope detector, we have a slow signal \( m(t) \) multiplied by a high speed signal \( \sin() \). So, so long as \( m(t) \geq 0 \) and envelope detector will connect the dots and recover \( m(t) \). If not, then we’ll recover \( |m(t)| \) as the envelope detector output.

(b) (25 points) Is \( r(t) \) an example of frequency modulation? What is the approximate bandwidth of \( r(t) \)?

SOLUTION: Carson’s rule \( 2(\beta + 1)B \) (or alternatively \( 2(\beta + 1)B \)) for the “carrier.” But the carrier is multiplied by \( m(t) \) which means convolution in frequency domain. This adds another \( 2B \) to the passband so the total bandwidth is \( 2(\beta + 2)B \).

3. (50 points) Cora, Marty and the FM Radio Station:

Cora the communications engineer has decided to open her own FM radio station. Unfortunately, the transmission tower is home to one Martin T. Sciuridae, a squirrel and Cora’s arch nemesis. Marty learns Cora is the new owner and sets out to sabotage her transmissions.

Assume Cora transmits

\[
r(t) = \cos \left( 2\pi f_c t - \beta \int_0^t m(\tau)d\tau \right)
\]

and that the transmission is wideband FM. Cora’s customers use the usual discriminators followed by envelope detectors as receivers.
Marty whips out a signal generator and corrupts \( r(t) \) as
\[
\ddot{r}(t) = \cos \left( 2\pi f_c t - \beta \int_0^t m(\tau) d\tau \right) - \left( \beta \int_0^t m(\tau) d\tau \right) \sin \left( 2\pi f_c t - \beta \int_0^t m(\tau) d\tau \right)
\]

(a) (10 points) Calculate the output of the differentiator in Cora’s listeners’ receivers.

**SOLUTION:**
\[
\frac{d\dot{r}(t)}{dt} = -(2\pi f_c - \beta m(t)) \sin(2\pi f_c t - \beta \int_0^t m(\tau) d\tau) - \beta m(t) \sin(2\pi f_c t - \beta \int_0^t m(\tau) d\tau) - \beta(2\pi f_c - \beta m(t)) (\int_0^t m(\tau) d\tau) \cos(2\pi f_c t - \beta \int_0^t m(\tau) d\tau)
\]
which reduces to
\[
\frac{d\dot{r}(t)}{dt} = -2\pi f_c \sin(2\pi f_c t - \beta \int_0^t m(\tau) d\tau) - \beta(2\pi f_c - \beta m(t)) (\int_0^t m(\tau) d\tau) \cos(2\pi f_c t - \beta \int_0^t m(\tau) d\tau)
\]

(b) (15 points) Does Marty’s devlish plan work? Why/why not?

**SOLUTION:** The differentiator is supposed to produce a high speed sinusoid modulated by the program material \( m(t) \). It does not, so Marty’s plan works.

(c) (15 points) Cora learns of Marty’s scheme and makes \( \beta \gg 2\pi f_c \). She then tells her listeners to modify their receivers by placing a hard limiter after the output of the differentiator to produce
\[
s(t) = \begin{cases} 1 & \frac{d\dot{r}(t)}{dt} > 0 \\ 0 & \text{o.w.} \end{cases}
\]
Rewrite \( s(t) \) in terms of \( \cos(2\pi f_c t - \beta \int_0^t m(\tau) d\tau) \).

**SOLUTION:** We’re left with
\[
\frac{d\ddot{r}(t)}{dt} = -2\pi f_c \sin(2\pi f_c t - \beta \int_0^t m(\tau) d\tau) - \beta(2\pi f_c - \beta m(t)) (\int_0^t m(\tau) d\tau) \cos(2\pi f_c t - \beta \int_0^t m(\tau) d\tau)
\]
If we assume \( \beta \) large, the first term is much smaller than the second so that
\[
\frac{d\ddot{r}(t)}{dt} \approx -\beta(2\pi f_c - \beta m(t)) \left( \int_0^t m(\tau) d\tau \right) \cos(2\pi f_c t - \beta \int_0^t m(\tau) d\tau)
\]
We can assume \( \beta(2\pi f_c - \beta m(t)) > 0 \) since \( f_c \) is assumed large. Then
\[
s(t) \approx \left| -\cos \left( 2\pi f_c t - \beta \int_0^t m(\tau) d\tau \right) \right|^+
\]
where (obviously)
\[
|x|^+ = \begin{cases} 1 & x > 0 \\ 0 & \text{o.w.} \end{cases}
\]
If we cannot assume \( \beta \) large, Cora’s almost certainly screwed.
(d) *(10 points)* Cora then asks her listeners to route \( s(t) \) into another FM receiver. Is Marty’s devilish plan foiled? Why/why not?

**HINT:** Where is the program material represented in an FM signal? Is that information preserved or corrupted by hard limiting?

**SOLUTION:** \( s(t) \) is a signal which bounces between 1 and 0 whose transition points are determined by the argument of the cosine. So, all the information about the program material is preserved.

We can show that pumping \( s(t) \) into an FM receiver gets you back the program material as follows. First

\[
\left| \frac{d|x(t)|}{dt} \right| = \sum_i \delta(t-t_i) \left| \frac{dx(t)}{dt} \right|
\]

where the \( t_i \) are the zero crossings of \( \cos(2\pi f_c t - \beta \int_0^t m(\tau) d\tau) \). Therefore,

\[
\left| \frac{d|s(t)|}{dt} \right| = \sum_i |(2\pi f_c - \beta m(t))| \delta(t-t_i)
\]

since at the zero crossings of \( \cos() \) we have \(|\sin()| = 1\).

Once again, we low pass filter (which is the second stage of an envelope detector) to connect the dots and out pops \(|2\pi f_c - \beta m(t)|\). And as usual, we get rid of the D.C. component \((2\pi f_c)\) by a high pass filter and obtain \(-\beta m(t)\) – which means Cora’s customers have recovered the program material.

How slick is that!?!?!