THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2008**
Quiz I

There are 3 questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Amplitude Modulation**

   (a) *(20 points)* Let $m(t) = e^{-t}u(t)$ where $u(t)$ is the unit step function. Let $r(t) = m(t)\cos 2\pi f_c t$ where $f_c \gg 1$. Provide an analytic expression for $R(f)$ the Fourier transform of $r(t)$ and carefully sketch $|R(f)|$.
   SOLUTION:
$$M(f) = \frac{1}{j2\pi f + 1}$$
$$R(f) = \frac{1}{2}M(f - f_c) + \frac{1}{2}M(f + f_c) = \frac{1}{2}\left[\frac{1}{j2\pi(f - f_c) + 1} + \frac{1}{j2\pi(f + f_c) + 1}\right]$$
$$|R(f)|^2 = \frac{1}{2}\left[\frac{1}{(2\pi(f - f_c))^2 + 1} + \frac{1}{(2\pi(f + f_c))^2 + 1}\right]$$
   Sketch: two humps, centered at f_c and $-f_c$.

   (b) *(10 points)* Carefully describe at least two different methods by which $m(t)$ can be recovered from $r(t)$. You must justify your answers.
   SOLUTION: *Bandwidth of signal is low comparied to f_c, so synchronous AM using $\cos 2\pi f_c t$ (multiplication followed by a low pass filter). Or since $m(t) > 0$ use an envelope detector.*

   (c) *(10 points)* Let
$$g(t) = \sqrt{\frac{1}{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}$$
   with Fourier transform
$$G(f) = e^{-2f^2\sigma^2}$$
   We form a periodic carrier signal
$$\phi(t) = \sum_{k=-\infty}^{\infty} g(t - \frac{k}{f_c})$$
   where f_c is the (fundamental) carrier frequency. Assume some general (i.e., different from the previous parts) program material $m(t)$ band limited to $B \ll f_c$. If $\sigma \ll \frac{1}{f_c}$, can $m(t)$ always be recovered from $r(t) = m(t)\phi(t)$ using an envelope detector? Carefully and analytically discuss why/why not.

SOLUTION: $\phi(t)$ *is a periodic sequence of pulses, so the Fourier transform of* $r(t)$ *will be a set of scaled replicas of* $M(f)$. *So, we could just low pass filter and get* $M(f)$ *back alone. However, we could also look at this another way. We've multiplied* $m(t)$ *by a set of non-negative "spikes" (the Gaussian function* $g(t)$ *looks more and more like an impulse, the smaller* σ *gets). So "connecting the dots" of* $m(t)\phi(t)$ *directly will give us back* $m(t)$. *HOWEVER, if* $m(t)$ *can be negative and we RECTIFY* $r(t)$*, we end up with* $|m(t)\phi(t)| = |m(t)|\phi(t)$ *and the envelope detector will recover* $|m(t)|$ *not* $m(t)$.

(d) *(10 points)* Suppose the diode in the envelope detector is faulty and functions as a simple resistor. Is the output of the envelope detector still $m(t)$? Always or never?

SOLUTION: *Now we don't rectify. The filter part of the envelope detector will "connect the dots" or "ski the peaks" and we'll obtain* $m(t)$ *directly – always.*

2. *(50 points)* **AM/FM/SchmeshM:**

Hermaphrodites Inc. has just introduced their new modulation system. They take band limited non-negative program material $m(t)$ and form the signal

$$r(t) = m(t)\sin\left(2\pi f_c t + \beta\int_0^t m(\tau)d\tau\right)$$

We assume that the bandwidth of $m(t)$ is $B \ll f_c$ and β is arbitrary so long as $\sin\left(2\pi f_c t + \beta\int_0^t m(\tau)d\tau\right)$ is always a high speed signal.

(a) *(25 points)* Is $r(t)$ an example of amplitude modulation? What is the output of an envelope detector with $r(t)$ as its input? Justify your answer.

SOLUTION: *As the title of the problem suggests, this could be both. The multiplier outside the sinusoid definitely modulates the high speed signal* $\sin()$. *However, the program material also appears in the argument of* $\sin()$. *Now as for the output of an envelope detector, we have a slow signal* $m(t)$ *multiplied by a high speed signal* $\sin()$. *So, so long as* $m(t) \geq 0$ *and envelope detector will connect the dots and recover* $m(t)$. *If not, then we'll recover* $|m(t)|$ *as the envelope detector output.*

(b) *(25 points)* Is $r(t)$ an example of frequency modulation? What is the approximate bandwidth of $r(t)$?

SOLUTION: *Carson's rule* $(2\beta+1)B$ *(or alternatively* $2(\beta+1)B$*) for the "carrier." But the carrier is multiplied by* $m(t)$ *which means convolution in frequency domain. This adds another* $2B$ *to the passband so the total bandwidth is* $2(\beta+2)B$.

3. *(50 points)* **Cora, Marty and the FM Radio Station:**

Cora the communications engineer has decided to open her own FM radio station. Unfortunately, the transmission tower is home to one Martin T. Sciuridae, a squirrel and Cora's arch nemisis. Marty learns Cora is the new owner and sets out to sabotage her transmissions.

Assume Cora transmits

$$r(t) = \cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)$$

and that the transmission is wideband FM. Cora's customers use the usual discriminators followed by envelope detectors as receivers.

Marty whips out a signal generator and corrupts $r(t)$ as

$$\tilde{r}(t) = \cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right) - \left(\beta\int_0^t m(\tau)d\tau\right)\sin\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)$$

(a) *(10 points)* Calculate the output of the differentiator in Cora's listeners' receivers.

SOLUTION:

$$\begin{aligned}\frac{d\tilde{r}(t)}{dt} &= -(2\pi f_c - \beta m(t))\sin\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\\ &- \beta m(t)\sin\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\\ &- \beta(2\pi f_c - \beta m(t))\left(\int_0^t m(\tau)d\tau\right)\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\end{aligned}$$

which reduces to

$$\begin{aligned}\frac{d\tilde{r}(t)}{dt} &= -2\pi f_c\sin\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\\ &- \beta(2\pi f_c - \beta m(t))\left(\int_0^t m(\tau)d\tau\right)\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\end{aligned}$$

(b) *(15 points)* Does Marty's devlish plan work? Why/why not?

SOLUTION: *The differentiator is supposed to produce a high speed sinusoid modulated by the program material $m(t)$. It does not, so Marty's plan works.*

(c) *(15 points)* Cora learns of Marty's scheme and makes $\beta \gg 2\pi f_c$. She then tells her listeners to modify their receivers by placing a hard limiter after the output of the differentiator to produce

$$s(t) = \begin{cases} 1 & \frac{d\tilde{r}(t)}{dt} > 0 \\ 0 & \text{o.w.} \end{cases}$$

Rewrite $s(t)$ in terms of $\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)$?

SOLUTION: *We're left with*

$$\begin{aligned}\frac{d\tilde{r}(t)}{dt} &= -2\pi f_c\sin\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\\ &- \beta(2\pi f_c - \beta m(t))\left(\int_0^t m(\tau)d\tau\right)\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\end{aligned}$$

If we assume β large, the first term is much smaller than the second so that

$$\frac{d\tilde{r}(t)}{dt} \approx -\beta(2\pi f_c - \beta m(t))\left(\int_0^t m(\tau)d\tau\right)\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)$$

We can sassume $\beta(2\pi f_c - \beta m(t)) > 0$ since f_c is assumed large. Then

$$s(t) \approx \left|-\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)\right|^+$$

where (obviously)

$$|x|^+ = \begin{cases} 1 & x > 0 \\ 0 & o.w. \end{cases}$$

If we cannot assume β large, Cora's almost certainly screwed.

(d) *(10 points)* Cora then asks her listeners to route $s(t)$ into another FM receiver. Is Marty's devilish plan foiled? Why/why not?

HINT: Where is the program material represented in an FM signal? Is that information preserved or corrupted by hard limiting?

SOLUTION: $s(t)$ *is a signal which bounces between* 1 *and* 0 *whose transition points are determined by the argument of the cosine. So, all the information about the program material is preserved.*

We can show that pumping $s(t)$ *into an FM receiver gets you back the program material as follows. First*

$$\left|\frac{d|x(t)|^+}{dt}\right| = \sum_i \delta(t-t_i)\left|\frac{dx(t)}{dt}\right|$$

where the t_i *are the zero crossings of* $\cos\left(2\pi f_c t - \beta\int_0^t m(\tau)d\tau\right)$. *Therefore,*

$$\left|\frac{d|s(t)|^+}{dt}\right| = \sum_i |(2\pi f_c - \beta m(t)|\,\delta(t-t_i)$$

since at the zero crossings of $\cos()$ *we have* $|\sin()| = 1$.

Once again, we low pass filter (which is the second stage of an envelope detector) to connect the dots and out pops $|2\pi f_c - \beta m(t)|$. *And as usual, we get rid of the D.C. component (*$2\pi f_c$*) by a high pass filter and obtain* $-\beta m(t)$ *– which means Cora's customers have recovered the program material.*

How slick is that!?!?!