Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Third Edition

Quiz Solutions

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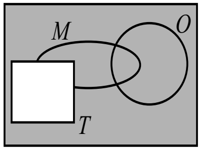
- The Matlab section quizzes at the end of each chapter use programs available for download as the archive `matcode.zip`. This archive has general purpose programs for solving probability problems as well as specific `.m` files associated with examples or quizzes in the text. Also available is a manual `probmatlab.pdf` describing the general purpose `.m` files in `matcode.zip`.

- We have made a substantial effort to check the solution to every quiz. Nevertheless, there is a nonzero probability (in fact, a probability close to unity) that errors will be found. If you find errors or have suggestions or comments, please send email to ***ryates@winlab.rutgers.edu***. When errors are found, corrected solutions will be posted at the website.

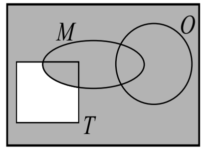
- This manual uses a page size matched to the screen of an iPad tablet. If you do print on paper and you have good eyesight, you may wish to print two pages per sheet in landscape mode. On the other hand, a "Fit to Paper" printing option will create "Large Print" output.

Quiz 1.1 Solution

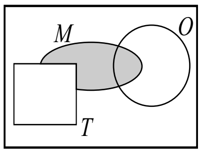
In the Venn diagrams for parts (a)-(g) below, the shaded area represents the indicated set.



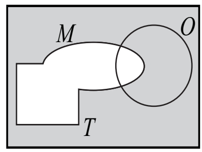
(a) $N = T^c$



(b) $N \cup M$



(c) $N \cap M$



(d) $T^c \cap M^c$

Quiz 1.2 Solution

$$
\begin{aligned}
A_1 &= \{vvv, vvd, dvv, dvd\} \\
B_1 &= \{vdv, vdd, ddv, ddd\} \\
A_2 &= \{vvv, ddd\} \\
B_2 &= \{vdv, dvd\} \\
A_3 &= \{vvv, vvd, vdv, dvv, vdd, dvd, ddv\} \\
B_3 &= \{ddd, ddv, dvd, vdd\}
\end{aligned}
$$

Recall that A_i and B_i are collectively exhaustive if $A_i \cup B_i = S$. Also, A_i and B_i are mutually exclusive if $A_i \cap B_i = \phi$. Since we have written down each pair A_i and B_i above, we can simply check for these properties.

The pair A_1 and B_1 are mutually exclusive and collectively exhaustive. The pair A_2 and B_2 are mutually exclusive but *not* collectively exhaustive. The pair A_3 and B_3 are not mutually exclusive since dvd belongs to A_3 and B_3. However, A_3 and B_3 are collectively exhaustive.

Quiz 1.3 Solution

There are exactly 50 equally likely outcomes: s_{51} through s_{100}. Each of these outcomes has probability $1/50$. It follows that

(a) $\mathrm{P}[\{s_{100}\}] = 1/50 = 0.02$.

(b) $\mathrm{P}[A] = \mathrm{P}[\{s_{90}, s_{91}, \ldots, s_{100}\}] = 11/50 = 0.22$.

(c) $\mathrm{P}[F] = \mathrm{P}[\{s_{51}, \ldots, s_{59}\}] = 9/50 = 0.18$.

(d) $\mathrm{P}[T < 90] = \mathrm{P}[\{s_{51}, \ldots, s_{89}\}] = 39/50. = 0.78$.

(e) $\mathrm{P}[C \text{ or better}] = \mathrm{P}[\{s_{70}, \ldots, s_{100}\}] = 31 \times 0.02 = 0.62$.

(f) $\mathrm{P}[\text{student passes}] = \mathrm{P}[\{s_{60}, \ldots, s_{100}\}] = 41 \times 0.02 = 0.82$.

Quiz 1.4 Solution

(a) The probability of exactly two voice packets is

$$\mathrm{P}\left[N_V = 2\right] = \mathrm{P}\left[\{vvd, vdv, dvv\}\right] = 0.3. \tag{1}$$

(b) The probability of at least one voice packet is

$$\begin{aligned}\mathrm{P}\left[N_V \geq 1\right] &= 1 - \mathrm{P}\left[N_V = 0\right] \\ &= 1 - \mathrm{P}\left[ddd\right] = 0.8.\end{aligned} \tag{2}$$

(c) The conditional probability of two voice packets followed by a data packet given that there were two voice packets is

$$\begin{aligned}\mathrm{P}\left[\{vvd\} \,|N_V = 2\right] &= \frac{\mathrm{P}\left[\{vvd\}, N_V = 2\right]}{\mathrm{P}\left[N_V = 2\right]} \\ &= \frac{\mathrm{P}\left[\{vvd\}\right]}{\mathrm{P}\left[N_V = 2\right]} = \frac{0.1}{0.3} = \frac{1}{3}.\end{aligned} \tag{3}$$

(d) The conditional probability of two data packets followed by a voice packet given there were two voice packets is

$$\mathrm{P}\left[\{ddv\} | N_V = 2\right] = \frac{\mathrm{P}\left[\{ddv\}, N_V = 2\right]}{\mathrm{P}\left[N_V = 2\right]} = 0.$$

The joint event of the outcome ddv and exactly two voice packets has probability zero since there is only one voice packet in the outcome ddv.

(e) The conditional probability of exactly two voice packets given at least one voice packet is

$$\begin{aligned}\mathrm{P}\left[N_V = 2 | N_v \geq 1\right] &= \frac{\mathrm{P}\left[N_V = 2, N_V \geq 1\right]}{\mathrm{P}\left[N_V \geq 1\right]} \\ &= \frac{\mathrm{P}\left[N_V = 2\right]}{\mathrm{P}\left[N_V \geq 1\right]} = \frac{0.3}{0.8} = \frac{3}{8}. \end{aligned} \tag{4}$$

(f) The conditional probability of at least one voice packet given there were exactly two voice packets is

$$\mathrm{P}\left[N_V \geq 1 | N_V = 2\right] = \frac{\mathrm{P}\left[N_V \geq 1, N_V = 2\right]}{\mathrm{P}\left[N_V = 2\right]} = \frac{\mathrm{P}\left[N_V = 2\right]}{\mathrm{P}\left[N_V = 2\right]} = 1. \tag{5}$$

Given two voice packets, there must have been at least one voice packet.

Quiz 1.5 Solution

We can describe this experiment by the event space consisting of the four possible events NL, NR, BL, and BR. We represent these events in the table:

	N	B
L	0.35	?
R	?	?

Once we fill in the table, finding the various probabilities will be simple.

In a roundabout way, the problem statement tells us how to fill in the table. In particular,

$$
\begin{aligned}
\mathrm{P}[N] &= 0.7 = \mathrm{P}[NL] + \mathrm{P}[NR], \\
\mathrm{P}[L] &= 0.6 = \mathrm{P}[NL] + \mathrm{P}[BL].
\end{aligned}
$$

Since $\mathrm{P}[NL] = 0.35$, we can conclude that $\mathrm{P}[NR] = 0.7 - 0.35 = 0.35$ and that $\mathrm{P}[BL] = 0.6 - 0.35 = 0.25$. This allows us to fill in two more table entries:

	N	B
L	0.35	0.25
R	0.35	?

The remaining table entry is filled in by observing that the probabilities must sum to 1. This implies $\mathrm{P}[BR] = 0.05$ and the complete table is

	N	B
L	0.35	0.25
R	0.35	0.05

The various probabilities are now simple:

(a) $$\begin{aligned} \mathrm{P}[B \cup L] &= \mathrm{P}[NL] + \mathrm{P}[BL] + \mathrm{P}[BR] \\ &= 0.35 + 0.25 + 0.05 = 0.65. \end{aligned}$$

(b) $$\begin{aligned} \mathrm{P}[N \cup L] &= \mathrm{P}[N] + \mathrm{P}[L] - \mathrm{P}[NL] \\ &= 0.7 + 0.6 - 0.35 = 0.95. \end{aligned}$$

(c) $\mathrm{P}[N \cup B] = \mathrm{P}[S] = 1.$

(d) $\mathrm{P}[LR] = \mathrm{P}[LL^c] = 0.$

Quiz 1.6 Solution

In this experiment, there are four outcomes with probabilities

$$\mathrm{P}[\{vv\}] = (0.8)^2 = 0.64, \qquad \mathrm{P}[\{vd\}] = (0.8)(0.2) = 0.16,$$
$$\mathrm{P}[\{dv\}] = (0.2)(0.8) = 0.16, \qquad \mathrm{P}[\{dd\}] = (0.2)^2 = 0.04.$$

When checking the independence of any two events A and B, it's wise to avoid intuition and simply check whether $\mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B]$. Using the probabilities of the outcomes, we now can test for the independence of events.

(a) First, we calculate the probability of the joint event:

$$\mathrm{P}\left[N_V = 2, N_V \geq 1\right] = \mathrm{P}\left[N_V = 2\right] = \mathrm{P}\left[\{vv\}\right] = 0.64. \tag{1}$$

Next, we observe that $\mathrm{P}[N_V \geq 1] = \mathrm{P}[\{vd, dv, vv\}] = 0.96.$. Finally, we make the comparison

$$\mathrm{P}\left[N_V = 2\right]\mathrm{P}\left[N_V \geq 1\right] = (0.64)(0.96) \neq \mathrm{P}\left[N_V = 2, N_V \geq 1\right], \tag{2}$$

which shows the two events are dependent.

(b) The probability of the joint event is

$$\mathrm{P}\left[N_V \geq 1, C_1 = v\right] = \mathrm{P}\left[\{vd, vv\}\right] = 0.80. \tag{3}$$

From part (a), $\mathrm{P}[N_V \geq 1] = 0.96$. Further, $\mathrm{P}[C_1 = v] = 0.8$ so that

$$\mathrm{P}\left[N_V \geq 1\right]\mathrm{P}\left[C_1 = v\right] = (0.96)(0.8) = 0.768 \neq \mathrm{P}\left[N_V \geq 1, C_1 = v\right]. \tag{4}$$

Hence, the events are dependent.

(c) The problem statement that the packets were independent implies that the events $\{C_2 = v\}$ and $\{C_1 = d\}$ are independent events. Just to be sure, we can do the calculations to check:

$$\mathrm{P}\left[C_1 = d, C_2 = v\right] = \mathrm{P}\left[\{dv\}\right] = 0.16. \tag{5}$$

Since $P[C_1 = d]\,P[C_2 = v] = (0.2)(0.8) = 0.16$, we confirm that the events are independent. Note that this shouldn't be surprising since we used the information that the packets were independent in the problem statement to determine the probabilities of the outcomes.

(d) The probability of the joint event is

$$P[C_2 = v, N_V \text{ is even}] = P[\{vv\}] = 0.64. \tag{6}$$

Also, each event has probability

$$P[C_2 = v] = P[\{dv, vv\}] = 0.8, \tag{7}$$
$$P[N_V \text{ is even}] = P[\{dd, vv\}] = 0.68. \tag{8}$$

Thus,

$$\begin{aligned} P[C_2 = v]\,P[N_V \text{ is even}] &= (0.8)(0.68) \\ &= 0.544 \neq P[C_2 = v, N_V \text{ is even}]. \end{aligned} \tag{9}$$

Thus the events are dependent.

Quiz 1.7 Solution

These two matlab instructions

```
>> T=randi(140,1000,5);
>> sum(T>120)
ans =
   126   147   134   133   163
```

simulate 5 runs of an experiment each with 1000 tweets. In particular, we note that `T=randi(140,1000,5)` generates a 1000×5 array `T` of pseudorandom integers between 1 and 140. Each column of `T` has 1000 entries representing an experimental run corresponding to the lengths of 1000 tweets. The comparison `T>120` produces a 5×1000 binary matrix in which each 1 marks a long tweet with length over 120 characters. Summing this binary array along the columns with the command `sum(T>120)` counts the number of long tweets in each experimental run.

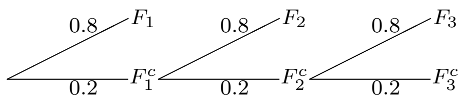
The experiment in which we examine the length of one tweet has sample space $S = \{s_1, s_2, \ldots, s_{140}\}$ with s_i denoting the outcome that a tweet has length i. Note that $\mathrm{P}[s_i] = 1/140$ and thus

$$\mathrm{P}\left[\text{tweet length} > 120\right] = \mathrm{P}\left[\{s_{121}, s_{122}, \ldots, s_{140}\}\right] = \frac{20}{140} = \frac{1}{7}. \tag{1}$$

Thus in each run of 1000 tweets, we would expect to see about 1/7 of the tweets, or about 143 tweets, to be be long tweets with length of over 120 characters. However, because the lengths are random, we see that we observe in the neighborhood of 143 long tweets in each run.

Quiz 2.1 Solution

Let F_i denote the event that that the user is found on page i. The tree for the experiment is



The user is found unless all three paging attempts fail. Thus the probability the user is found is

$$\mathrm{P}[F] = 1 - \mathrm{P}[F_1^c F_2^c F_3^c] = 1 - (0.2)^3 = 0.992. \tag{1}$$

Quiz 2.2 Solution

(a) We can view choosing each bit in the code word as a subexperiment. Each subexperiment has two possible outcomes: 0 and 1. Thus by the fundamental principle of counting, there are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ possible code words.

(b) An experiment that can yield all possible code words with two zeroes is to choose which 2 bits (out of 4 bits) will be zero. The other two bits then must be ones. There are $\binom{4}{2} = 6$ ways to do this. Hence, there are six code words with exactly two zeroes. For this problem, it is also possible to simply enumerate the six code words:

$$\begin{array}{ccc} 1100, & 1010, & 1001, \\ 0101, & 0110, & 0011. \end{array}$$

(c) When the first bit must be a zero, then the first subexperiment of choosing the first bit has only one outcome. For each of the next three bits, we have two choices. In this case, there are $1 \times 2 \times 2 \times 2 = 8$ ways of choosing a code word.

(d) For the constant ratio code, we can specify a code word by choosing M of the bits to be ones. The other $N-M$ bits will be zeroes. The number of ways of choosing such a code word is $\binom{N}{M}$. For $N=8$ and $M=3$, there are $\binom{8}{3}=56$ code words.

Quiz 2.3 Solution

(a) In this problem, k bits received in error is the same as k failures in 100 trials. The failure probability is $\epsilon = 1-p$ and the success probability is $1-\epsilon = p$. That is, the probability of k bits in error and $100-k$ correctly received bits is

$$P[E_{k,100-k}] = \binom{100}{k}\epsilon^k(1-\epsilon)^{100-k}. \tag{1}$$

For $\epsilon = 0.01$,

$$P[E_{0,100}] = (1-\epsilon)^{100} = (0.99)^{100} = 0.3660. \tag{2}$$
$$P[E_{1,99}] = 100(0.01)(0.99)^{99} = 0.3700. \tag{3}$$
$$P[E_{2,98}] = 4950(0.01)^2(0.99)^{98} = 0.1849. \tag{4}$$
$$P[E_{3,97}] = 161,700(0.01)^3(0.99)^{97} = 0.0610. \tag{5}$$

(b) The probability a packet is decoded correctly is just

$$P[C] = P[E_{0,100}] + P[E_{1,99}] + P[E_{2,98}] + P[E_{3,97}] = 0.9819. \tag{6}$$

Quiz 2.4 Solution

(a) Since the chip works only if all n transistors work, the transistors in the chip are like devices in series. The probability that a chip works is $P[C] = p^n$.

(b) The module works if either 8 chips work or 9 chips work. Let C_k denote the event that exactly k chips work. Since transistor failures are independent of each other, chip failures are also independent. Thus each $P[C_k]$ has the binomial probability

$$P[C_8] = \binom{9}{8} (P[C])^8 (1 - P[C])^{9-8} = 9p^{8n}(1 - p^n), \tag{1}$$

$$P[C_9] = (P[C])^9 = p^{9n}. \tag{2}$$

The probability a memory module works is

$$P[M] = P[C_8] + P[C_9] = p^{8n}(9 - 8p^n). \tag{3}$$

(c) Given that $p = 0.999$. For and we need to find the largest value of n such that $P[M] > 0.9$. Although this quiz is not a MATLAB quiz, this matlab script is an easy way to calculate the largest n:

```
%chipsize1.m
n=1:80;
PM=(p.^(8*n)).*(9-8*(p.^n));
plot(n,PM)
nmax = sum(PM>0.9)
```

The script includes a plot command to verify that $P[M]$ is a decreasing function of n. The output is

```
>> chipsize1
nmax =
    62
```

(d) Now the event C_7 that seven chips works also yields an acceptable module. Since each chip works with probability $P[C] = p^n$,

$$\begin{aligned} P[C_7] = \binom{9}{7} (P[C])^7 (1 - P[C])^2 &= 36p^{7n}(1 - p^n)^2 \\ &= 36p^{7n} - 72p^{8n} + 36p^{9n}. \end{aligned} \tag{4}$$

The probability a memory module works is

$$
\begin{aligned}
\mathrm{P}[M] &= \mathrm{P}[C_7] + \mathrm{P}[C_8] + \mathrm{P}[C_9] \\
&= 36p^{7n} - 72p^{8n} + 36p^{9n} + p^{8n}(9 - 8p^n) && (5) \\
&= 36p^{7n} - 63p^{8n} + 28p^{9n}. && (6)
\end{aligned}
$$

Just as we did in the previous part, we use MATLAB to find the maximum n:

```
%chipsize2.m
n=1:150;
PM=36*(p.^(7*n))-(63*p.^(8*n))+(28*p.^(9*n));
plot(n,PM)
nmax = sum(PM>0.9)
```

The answer is

```
>> chipsize2
nmax =
   138
```

The additional redundancy at the chip level to enable one more defective chip allows us to more than double the number of transistors per chip.

Quiz 2.5 Solution

For a MATLAB simulation, we first generate a vector `R` of 100 random numbers. Second, we generate vector `X` as a function of `R` to represent the 3 possible outcomes of a flip. That is, `X(i)=1` if flip i was heads, `X(i)=2` if flip i was tails, and `X(i)=3)` is flip i landed on the edge. The matlab code is

```
R=rand(1,100);
X=(R<= 0.4) ...
    + (2*(R>0.4).*(R<=0.9)) ...
    + (3*(R>0.9));
Y=hist(X,1:3)
```

To see how this works, we note there are three cases:

- If `R(i) <= 0.4`, then `X(i)=1`.
- If `0.4 < R(i)` and `R(i)<=0.9`, then `X(i)=2`.
- If `0.9 < R(i)`, then `X(i)=3`.

These three cases will have probabilities 0.4, 0.5 and 0.1. Lastly, we use the **`hist`** function to count how many occurences of each possible value of `X(i)`.

Quiz 3.1 Solution

The sample space, probabilities and corresponding grades for the experiment are

Outcomes	BB	BC	CB	CC
G_2	3.0	2.5	2.5	2.0

Quiz 3.2 Solution

(a) To find c, we recall that the PMF must sum to 1. That is,

$$\sum_{n=1}^{3} P_N(n) = c\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1. \tag{1}$$

This implies $c = 6/11$. Now that we have found c, the remaining parts are straightforward.

(b) $\mathrm{P}[N = 1] = P_N(1) = c = 6/11.$

(c) $$\begin{aligned}\mathrm{P}[N \geq 2] &= P_N(2) + P_N(3)\\ &= c/2 + c/3 = 5/11.\end{aligned}$$

(d) $\mathrm{P}[N > 3] = \sum_{n=4}^{\infty} P_N(n) = 0.$

Quiz 3.3 Solution

Decoding each transmitted bit is an independent trial where we call a bit error a "success." Each bit is in error, that is, the trial is a success, with probability p. Now we can interpret each experiment in the generic context of independent trials.

(a) The random variable X is the number of trials up to and including the first success. Similar to Example 3.9, X has the geometric PMF

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $p = 0.1$, then the probability exactly 10 bits are sent is

$$P_X(10) = (0.1)(0.9)^9 = 0.0387. \tag{1}$$

The probability that at least 10 bits are sent is

$$\mathrm{P}[X \geq 10] = \sum_{x=10}^{\infty} P_X(x). \tag{2}$$

This sum is not too hard to calculate. However, its even easier to observe that $X \geq 10$ if the first 10 bits are transmitted correctly. That is,

$$\mathrm{P}[X \geq 10] = \mathrm{P}[\text{first 10 bits correct}] = (1-p)^{10}. \tag{3}$$

For $p = 0.1$,

$$\mathrm{P}[X \geq 10] = 0.9^{10} = 0.3487. \tag{4}$$

(c) The random variable Y is the number of successes in 100 independent trials. Just as in Example 3.11, Y has the binomial PMF

$$P_Y(y) = \binom{100}{y} p^y (1-p)^{100-y}. \tag{5}$$

If $p = 0.01$, the probability of exactly 2 errors is

$$P_Y(2) = \binom{100}{2} (0.01)^2 (0.99)^{98} = 0.1849. \tag{6}$$

(d) The probability of no more than 2 errors is

$$\begin{aligned} \mathrm{P}[Y \leq 2] &= P_Y(0) + P_Y(1) + P_Y(2) \\ &= (0.99)^{100} + 100(0.01)(0.99)^{99} + \binom{100}{2}(0.01)^2(0.99)^{98} \\ &= 0.9207. \end{aligned} \tag{7}$$

(e) Random variable Z is the number of trials up to and including the third success. Thus Z has the Pascal PMF (see Example 3.13)

$$P_Z(z) = \binom{z-1}{2} p^3 (1-p)^{z-3}. \tag{8}$$

Note that $P_Z(z) > 0$ for $z = 3, 4, 5, \ldots$.

(f) If $p = 0.25$, the probability that the third error occurs on bit 12 is

$$P_Z(12) = \binom{11}{2} (0.25)^3 (0.75)^9 = 0.0645. \tag{9}$$

Quiz 3.4 Solution

Each of these probabilities can be read from the graph of the CDF $F_Y(y)$. However, we must keep in mind that when $F_Y(y)$ has a discontinuity at y_0, $F_Y(y)$ takes the upper value $F_Y(y_0^+)$.

(a) $P[Y < 1] = F_Y(1^-) = 0$.

(b) $P[Y \leq 1] = F_Y(1) = 0.6$.

(c) $P[Y > 2] = 1 - P[Y \leq 2] = 1 - F_Y(2) = 1 - 0.8 = 0.2$.

(d) $P[Y \geq 2] = 1 - P[Y < 2] = 1 - F_Y(2^-) = 1 - 0.6 = 0.4$.

(e) $P[Y = 1] = P[Y \leq 1] - P[Y < 1] = F_Y(1^+) - F_Y(1^-) = 0.6$.

(f) $P[Y = 3] = P[Y \leq 3] - P[Y < 3] = F_Y(3^+) - F_Y(3^-) = 0.8 - 0.8 = 0$.

Quiz 3.5 Solution

(a) With probability $1/3$, the subscriber sends a text and the cost is $C = 10$ cents. Otherwise, with probability $2/3$, the subscriber receives a text and the cost is $C = 5$ cents. This corresponds to the PMF

$$P_C(c) = \begin{cases} 2/3 & c = 5, \\ 1/3 & c = 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The expected value of C is

$$\mathrm{E}[C] = (2/3)(5) + (1/3)(10) = 6.67 \text{ cents.} \tag{2}$$

(c) For the next two parts we think of each text as a Bernoulli trial such that the trial is a "success" if the subscriber sends a text. The success probability is $p = 1/3$. Let R denote the number of texts received before sending a text. In terms of Bernoulli trials, R is the number of failures before the first success. R is similar to a geometric random variable except $R = 0$ is possible if the first text is sent rather than received. In general $R = r$ if the first r trials are failures (i.e. the first r texts are received) and trial $r + 1$ is a success. Thus R has PMF

$$P_R(r) = \begin{cases} (1-p)^r p & r = 0, 1, 2 \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The probability of receiving four texts before sending a text is

$$P_R(4) = (1-p)^4 p. \tag{4}$$

(d) The expected number of texts received before sending a text is

$$\mathrm{E}[R] = \sum_{r=0}^{\infty} r P_R(r) = \sum_{r=0}^{\infty} r(1-p)^r p. \tag{5}$$

Letting $q = 1 - p$ and observing that the $r = 0$ term in the sum is zero,

$$\mathrm{E}[R] = p \sum_{r=1}^{\infty} r q^r. \tag{6}$$

Using Math Fact B.7, we have

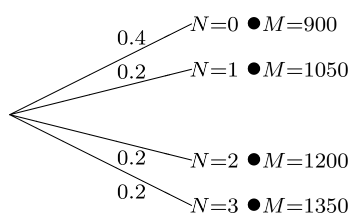
$$\mathrm{E}[R] = p \frac{q}{(1-q)^2} = \frac{1-p}{p} = 2. \tag{7}$$

Quiz 3.6 Solution

(a) As a function of N, the money spent by the three customers is

$$M = 450N + 300(3 - N) = 900 + 150N.$$

(b) To find the PMF of M, we can draw the following tree and map the outcomes to values of M:



From this tree,

$$P_M(m) = \begin{cases} 0.4 & m = 900, \\ 0.2 & m = 1050, 1200, 1350 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

From the PMF $P_M(m)$, the expected value of M is

$$\begin{aligned} \mathrm{E}[M] &= 900P_M(900) + 1050P_M(1050) \\ &\quad + 1200P_M(1200) + 1350P_M(1350) \qquad (2) \\ &= (900)(0.4) + (1050 + 1200 + 1350)(0.2) = 1080. \qquad (3) \end{aligned}$$

Quiz 3.7 Solution

(a) Using Definition 3.13, the expected number of applications is

$$\begin{aligned} \mathrm{E}[A] &= \sum_{a=1}^{4} aP_A(a) \\ &= 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) \\ &= 2. \end{aligned} \qquad (1)$$

(b) The number of memory chips is

$$M = g(A) = \begin{cases} 4 & A = 1, 2, \\ 6 & A = 3, \\ 8 & A = 4. \end{cases} \qquad (2)$$

(c) By Theorem 3.10, the expected number of memory chips is

$$\begin{aligned} \mathrm{E}[M] &= \sum_{a=1}^{4} g(A)P_A(a) \\ &= 4(0.4) + 4(0.3) + 6(0.2) + 8(0.1) \\ &= 4.8. \end{aligned} \qquad (3)$$

Since $\mathrm{E}[A] = 2$,

$$g(\mathrm{E}[A]) = g(2) = 4.$$

However, $\mathrm{E}[M] = 4.8 \neq g(\mathrm{E}[A])$. The two quantities are different because $g(A)$ is not of the form $\alpha A + \beta$.

Quiz 3.8 Solution

For this problem, it is helpful to wrote out the PMF of N in the table

n	0	1	2	3
$P_N(n)$	0.4	0.3	0.2	0.1

The PMF $P_N(n)$ allows us to calculate each of the desired quantities.

(a) The expected value is

$$\begin{aligned} \mathrm{E}[N] &= \sum_{n=0}^{3} n P_N(n) \\ &= 0(0.4) + 1(0.3) + 2(0.2) + 3(0.1) = 1. \end{aligned} \tag{1}$$

(b) The second moment of N is

$$\begin{aligned} \mathrm{E}\left[N^2\right] &= \sum_{n=0}^{3} n^2 P_N(n) \\ &= 0^2(0.4) + 1^2(0.3) + 2^2(0.2) + 3^2(0.1) = 2. \end{aligned} \tag{2}$$

(c) The variance of N is

$$\mathrm{Var}[N] = \mathrm{E}\left[N^2\right] - (\mathrm{E}[N])^2 = 2 - 1^2 = 1. \tag{3}$$

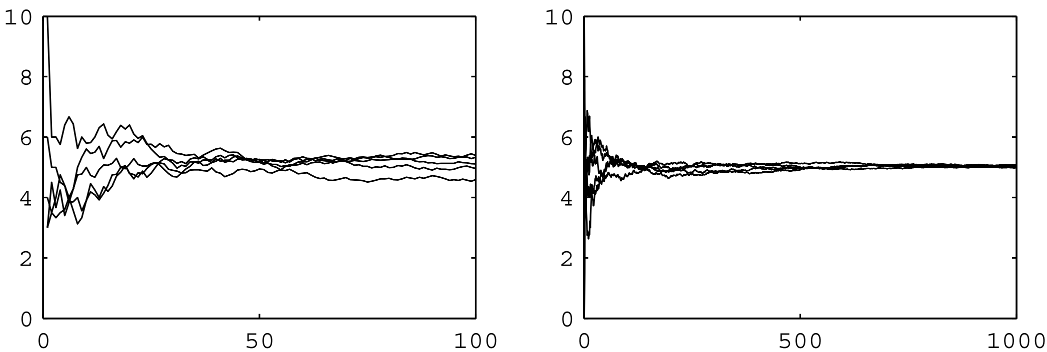
(d) The standard deviation is $\sigma_N = \sqrt{\mathrm{Var}[N]} = 1$.

Quiz 3.9 Solution

The function `samplemean(k)` generates and plots five m_n sequences for $n = 1, 2, \ldots, k$. The ith column `M(:,i)` of `M` holds a sequence $m_1, m_2, \ldots, m_k$.

```
function M=samplemean(k);
K=(1:k)';
M=zeros(k,5);
for i=1:5,
    X=duniformrv(0,10,k);
    M(:,i)=cumsum(X)./K;
end;
plot(K,M);
```

Here are two examples of `samplemean`:

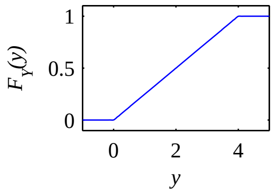


(a) `samplemean(100)` **(b)** `samplemean(1000)`

Each time `samplemean(k)` is called produces a random output. What is observed in these figures is that for small n, m_n is fairly random but as n gets large, m_n gets close to $\mathrm{E}[X] = 5$. Although each sequence $m_1, m_2, \ldots$ that we generate is random, the sequences always converges to $\mathrm{E}[X]$. This random convergence is analyzed in Chapter 10.

Quiz 4.2 Solution

The CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \le y \le 4, \\ 1 & y > 4. \end{cases} \tag{1}$$

From the CDF $F_Y(y)$, we can calculate the probabilities:

(a) $P[Y \le -1] = F_Y(-1) = 0$

(b) $P[Y \le 1] = F_Y(1) = 1/4$

(c)
$$\begin{aligned} P[2 < Y \le 3] &= F_Y(3) - F_Y(2) \\ &= 3/4 - 2/4 = 1/4. \end{aligned}$$

(d)
$$\begin{aligned} P[Y > 1.5] &= 1 - P[Y \le 1.5] \\ &= 1 - F_Y(1.5) \\ &= 1 - (1.5)/4 = 5/8. \end{aligned}$$

Quiz 4.3 Solution

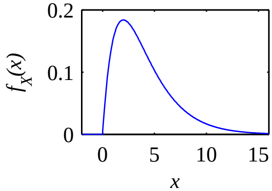
(a) First we will find the constant c and then we will sketch the PDF. To find c, we use the fact that

$$1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^{\infty} cxe^{-x/2}\, dx. \tag{1}$$

We evaluate this integral using integration by parts:

$$\begin{aligned} 1 &= \underbrace{-2cxe^{-x/2}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} 2ce^{-x/2}\, dx \\ &= -4ce^{-x/2}\Big|_0^{\infty} = 4c. \end{aligned} \tag{2}$$

Thus $c = 1/4$ and X has the Erlang $(n = 2, \lambda = 1/2)$ PDF

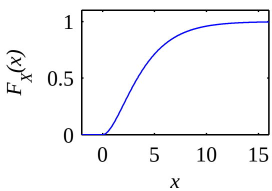


$$f_X(x) = \begin{cases} (x/4)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) To find the CDF $F_X(x)$, we first note X is a nonnegative random variable so that $F_X(x) = 0$ for all $x < 0$. For $x \geq 0$,

$$\begin{aligned} F_X(x) = \int_0^x f_X(y)\,dy &= \int_0^x \frac{y}{4} e^{-y/2}\,dy \\ &= -\frac{y}{2} e^{-y/2}\Big|_0^x + \int_0^x \frac{1}{2} e^{-y/2}\,dy \\ &= 1 - \frac{x}{2} e^{-x/2} - e^{-x/2}. \end{aligned} \tag{3}$$

The complete expression for the CDF is



$$F_X(x) = \begin{cases} 1 - \left(\frac{x}{2} + 1\right) e^{-x/2} & x \geq 0, \\ 0 & \text{ow.} \end{cases}$$

(c) From the CDF $F_X(x)$,

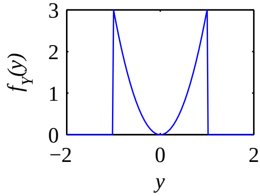
$$\begin{aligned} P[0 \leq X \leq 4] &= F_X(4) - F_X(0) \\ &= 1 - 3e^{-2}. \end{aligned} \tag{4}$$

(d) Similarly,

$$\begin{aligned} P[-2 \leq X \leq 2] &= F_X(2) - F_X(-2) \\ &= 1 - 3e^{-1}. \end{aligned} \tag{5}$$

Quiz 4.4 Solution

The PDF of Y is



$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The expected value of Y is

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-1}^{1} (3/2)y^3\, dy = (3/8)y^4\Big|_{-1}^{1} = 0. \tag{2}$$

Note that the above calculation wasn't really necessary because $\mathrm{E}[Y] = 0$ whenever the PDF $f_Y(y)$ is an even function, i.e., $f_Y(y) = f_Y(-y)$.

(b) The second moment of Y is

$$\mathrm{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_{-1}^{1} (3/2)y^4\, dy = (3/10)y^5\Big|_{-1}^{1} = 3/5. \tag{3}$$

(c) The variance of Y is

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 = 3/5. \tag{4}$$

(d) The standard deviation of Y is $\sigma_Y = \sqrt{\mathrm{Var}[Y]} = \sqrt{3/5}$.

Quiz 4.5 Solution

(a) When X is an exponential (λ) random variable, $\mathrm{E}[X] = 1/\lambda$ and $\mathrm{Var}[X] = 1/\lambda^2$. Since $\mathrm{E}[X] = 3$ and $\mathrm{Var}[X] = 9$, we must have $\lambda = 1/3$. The PDF of X is

$$f_X(x) = \begin{cases} (1/3)e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) We know X is a uniform (a, b) random variable. To find a and b, we apply Theorem 4.6 to write

$$\mathrm{E}[X] = \frac{a+b}{2} = 3 \tag{2}$$

$$\mathrm{Var}[X] = \frac{(b-a)^2}{12} = 9. \tag{3}$$

This implies

$$a + b = 6, \qquad b - a = \pm 6\sqrt{3}. \tag{4}$$

The only valid solution with $a < b$ is

$$a = 3 - 3\sqrt{3}, \qquad b = 3 + 3\sqrt{3}. \tag{5}$$

The complete expression for the PDF of X is

$$f_X(x) = \begin{cases} 1/(6\sqrt{3}) & 3 - 3\sqrt{3} < x < 3 + 3\sqrt{3}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

(c) We know that the Erlang (n, λ) random variable has PDF

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The expected value and variance are $\mathrm{E}[X] = n/\lambda$ and $\mathrm{Var}[X] = n/\lambda^2$. This implies

$$\frac{n}{\lambda} = 3, \qquad \frac{n}{\lambda^2} = 9. \tag{8}$$

It follows that

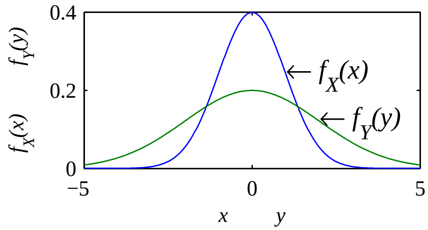
$$n = 3\lambda = 9\lambda^2. \tag{9}$$

Thus $\lambda = 1/3$ and $n = 1$. As a result, the Erlang (n, λ) random variable must be the exponential $(\lambda = 1/3)$ random variable with PDF

$$f_X(x) = \begin{cases} (1/3)e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Quiz 4.6 Solution

The PDFs of X and Y are:



The fact that Y has twice the standard deviation of X is reflected in the greater spread of $f_Y(y)$. However, it is important to remember that as the standard deviation increases, the peak value of the Gaussian PDF goes down.

Each of the requested probabilities can be calculated using $\Phi(z)$ function and Table 4.2 or $Q(z)$ and Table 4.3.

(a) Since X is Gaussian $(0, 1)$,

$$\begin{aligned} \mathrm{P}[-1 < X \leq 1] &= F_X(1) - F_X(-1) \\ &= \Phi(1) - \Phi(-1) \\ &= 2\Phi(1) - 1 \\ &= 0.6826. \end{aligned} \tag{1}$$

(b) Since Y is Gaussian $(0, 2)$,

$$\begin{aligned} \mathrm{P}[-1 < Y \leq 1] &= F_Y(1) - F_Y(-1) \\ &= \Phi\left(\frac{1}{\sigma_Y}\right) - \Phi\left(\frac{-1}{\sigma_Y}\right) \\ &= 2\Phi\left(\frac{1}{2}\right) - 1 = 0.383. \end{aligned} \tag{2}$$

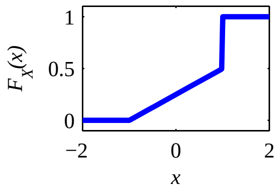
(c) Again, since X is Gaussian $(0, 1)$, $\mathrm{P}[X > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$.

(d) Since Y is Gaussian $(0, 2)$,

$$P[Y > 3.5] = Q\left(\frac{3.5}{2}\right) = 1 - \Phi(1.75) = 0.04. \qquad (3)$$

Quiz 4.7 Solution

The CDF of X is



$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \leq x < 1, \\ 1 & x \geq 1. \end{cases} \qquad (1)$$

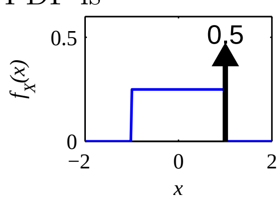
The following probabilities can be read directly from the CDF:

(a) $P[X \leq 1] = F_X(1) = 1$.

(b) $P[X < 1] = F_X(1^-) = 1/2$.

(c) $P[X = 1] = F_X(1^+) - F_X(1^-) = 1/2$.

(d) We find the PDF $f_Y(y)$ by taking the derivative of $F_Y(y)$. The resulting PDF is



$$f_X(x) = \begin{cases} \frac{1}{4} & -1 \leq x < 1, \\ \frac{\delta(x-1)}{2} & x = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Quiz 4.8 Solution

A natural way to produce random variables with PDF $f_{T|T>2}(t)$ is to generate samples of T with PDF $f_T(t)$ and then to discard those samples which fail to satisfy the condition $T > 2$. Here is a MATLAB function that uses this method:

```
function t=t2rv(m)
i=0;lambda=1/3;
t=zeros(m,1);
while (i<m),
    x=exponentialrv(lambda,1);
    if (x>2)
        t(i+1)=x;
        i=i+1;
    end
end
```

A second method exploits the fact that if T is an exponential (λ) random variable, then $T' = T + 2$ has PDF $f_{T'}(t) = f_{T|T>2}(t)$. In this case the command

```
t=2.0+exponentialrv(1/3,m)
```

generates the vector `t`.

Quiz 5.1 Solution

Each value of the joint CDF can be found by considering the corresponding probability.

(a) $F_{X,Y}(-\infty, 2) = P[X \leq -\infty, Y \leq 2] \leq P[X \leq -\infty] = 0$ since X cannot take on the value $-\infty$.

(b) $F_{X,Y}(\infty, \infty) = P[X \leq \infty, Y \leq \infty] = 1$.

This result is given in Theorem 5.1.

(c) $F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y)$.

(d) $F_{X,Y}(\infty, -\infty) = P[X \leq \infty, Y \leq -\infty] = P[Y \leq -\infty] = 0$ since Y cannot take on the value $-\infty$.

Quiz 5.2 Solution

From the joint PMF of Q and G given in the table, we can calculate the requested probabilities by summing the PMF over those values of Q and G that correspond to the event.

(a) The probability that $Q = 0$ is

$$\begin{aligned} P[Q = 0] &= P_{Q,G}(0,0) + P_{Q,G}(0,1) + P_{Q,G}(0,2) + P_{Q,G}(0,3) \\ &= 0.06 + 0.18 + 0.24 + 0.12 = 0.6. \end{aligned} \tag{1}$$

(b) The probability that $Q = G$ is

$$P[Q = G] = P_{Q,G}(0,0) + P_{Q,G}(1,1) = 0.18. \tag{2}$$

(c) The probability that $G > 1$ is

$$\begin{aligned} P[G > 1] &= \sum_{g=2}^{3} \sum_{q=0}^{1} P_{Q,G}(q,g) \\ &= 0.24 + 0.16 + 0.12 + 0.08 = 0.6. \end{aligned} \tag{3}$$

(d) The probability that $G > Q$ is

$$\begin{aligned} P[G > Q] &= \sum_{q=0}^{1} \sum_{g=q+1}^{3} P_{Q,G}(q,g) \\ &= 0.18 + 0.24 + 0.12 + 0.16 + 0.08 = 0.78. \end{aligned} \tag{4}$$

Quiz 5.3 Solution

By Theorem 5.4, the marginal PMF of H is

$$P_H(h) = \sum_{b=0,2,4} P_{H,B}(h,b). \tag{1}$$

For each value of h, this corresponds to calculating the row sum across the table of the joint PMF. Similarly, the marginal PMF of B is

$$P_B(b) = \sum_{h=-1}^{1} P_{H,B}(h,b). \tag{2}$$

For each value of b, this corresponds to the column sum down the table of the joint PMF. The easiest way to calculate these marginal PMFs is to simply sum each row and column:

$P_{H,B}(h,b)$	$b=0$	$b=2$	$b=4$	$P_H(h)$
$h=-1$	0	0.4	0.2	0.6
$h=0$	0.1	0	0.1	0.2
$h=1$	0.1	0.1	0	0.2
$P_B(b)$	0.2	0.5	0.3	

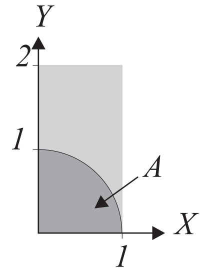
Quiz 5.4 Solution

To find the constant c, we apply

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy \\ &= \int_0^2 \int_0^1 cxy\, dx\, dy = c \int_0^2 y \left(\left. \frac{x^2}{2} \right|_0^1 \right) dy = \frac{c}{2} \int_0^2 y\, dy = \left. \frac{cy^2}{4} \right|_0^2 = c. \end{aligned} \tag{1}$$

Thus $c = 1$.
To calculate P[A], we write

$$\mathrm{P}[A] = \iint_A f_{X,Y}(x,y)\, dx\, dy \tag{2}$$



To integrate over A, we convert to polar coordinates using the substitutions $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.
This yields

$$\begin{aligned}
\mathrm{P}[A] &= \int_0^{\pi/2}\int_0^1 r^2 \sin\theta\cos\theta\, r\, dr\, d\theta \\
&= \left(\int_0^1 r^3\, dr\right)\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \left(r^4/4\Big|_0^1\right)\left(\frac{\sin^2\theta}{2}\Bigg|_0^{\pi/2}\right) = \frac{1}{8}. \qquad (3)
\end{aligned}$$

Quiz 5.5 Solution

By Theorem 5.8, the marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \tag{1}$$

Note that $f_X(x) = 0$ for $x < 0$ or $x > 1$. For $0 \le x \le 1$,

$$f_X(x) = \frac{6}{5}\int_0^1 (x + y^2)\, dy = \frac{6}{5}\left(xy + y^3/3\right)\Big|_{y=0}^{y=1} = \frac{6x+2}{5} \tag{2}$$

The complete expression for the PDF of X is

$$f_X(x) = \begin{cases} (6x+2)/5 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

By the same method we obtain the marginal PDF for Y. For $0 \le y \le 1$,

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \\
&= \frac{6}{5}\int_0^1 (x + y^2)\, dx = \frac{6}{5}\left(\frac{x^2}{2} + xy^2\right)\Bigg|_{x=0}^{x=1} = \frac{6y^2+3}{5}. \qquad (4)
\end{aligned}$$

Since $f_Y(y) = 0$ for $y < 0$ or $y > 1$, the complete expression for the PDF of Y is

$$f_Y(y) = \begin{cases} (3+6y^2)/5 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Quiz 5.6(A) Solution

(a) For random variables X and Y from Example 5.3, we observe that $P_Y(1) = 0.09$ and $P_X(0) = 0.01$. However,

$$P_{X,Y}(0,1) = 0 \neq P_X(0)\, P_Y(1) \tag{1}$$

Since we have found a pair x, y such that $P_{X,Y}(x,y) \neq P_X(x)P_Y(y)$, we can conclude that X and Y are dependent. Note that whenever $P_{X,Y}(x,y) = 0$, independence requires that either $P_X(x) = 0$ or $P_Y(y) = 0$.

(b) For random variables Q and G from Quiz 5.2, it is not obvious whether they are independent. Unlike X and Y in part (a), there are no obvious pairs q, g that fail the independence requirement. In this case, we calculate the marginal PMFs from the table of the joint PMF $P_{Q,G}(q,g)$ in Quiz 5.2. In transposed form, this table is

$P_{Q,G}(q,g)$	$q=0$	$q=1$	$P_G(g)$
$g=0$	0.06	0.04	0.10
$g=1$	0.18	0.12	0.30
$g=2$	0.24	0.16	0.40
$g=3$	0.12	0.08	0.20
$P_Q(q)$	0.60	0.40	

Careful study of the table will verify that $P_{Q,G}(q,g) = P_Q(q)P_G(g)$ for every pair q, g. Hence Q and G are independent.

Quiz 5.6(B) Solution

Since X_1 and X_2 are identical, $f_{X_1}(x) = f_{X_2}(x) = f_X(x)$. Since X_1 and X_2 are independent,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2) = \begin{cases} \frac{x_1}{2} \cdot \frac{x_2}{2} & 0 \leq x_1, x_2 \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Quiz 5.7 Solution

You may have noticed that this quiz was inadvertently missing in the book. Here is the missing quiz:

> **Quiz 5.7:** The company's cost C of transmitting a page depends on page length L (in kilobytes) and transmission rate R (in Mb/s). Specifically, $C = h(L, R) = 0.001L + 0.1R$ cents. Just as in Example 5.15, the probability model is that L and R are independent with PMFs
>
> $$P_R(r) = \begin{cases} 0.4 & r = 5, \\ 0.6 & r = 10, \\ 0 & \text{otherwise,} \end{cases} \qquad P_L(l) = \begin{cases} 0.3 & l = 750, \\ 0.5 & l = 1500, \\ 0.2 & l = 2500, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$
>
> Derive the expected cost $\mathrm{E}[C] = \mathrm{E}[h(L, R)]$.
> Does $\mathrm{E}[(h(L, R)] = h(\mathrm{E}[L], \mathrm{E}[R])$?

Quiz Solution 5.7: Since L and R are independent,

$$\begin{aligned} \mathrm{E}\left[h(L, R)\right] &= \sum_l \sum_r P_L(l)\, P_R(r)\, h(l, r) \\ &= \sum_l \sum_r P_L(l)\, P_R(r)\, (0.001l + 0.1r) \\ &= \sum_l \sum_r P_L(l)\, P_R(r)\, 0.001l + \sum_l \sum_r P_L(l)\, P_R(r)\, 0.1r. \end{aligned} \tag{2}$$

In each double summation, the sums over l and r can be separated, yielding

$$\begin{aligned}
\mathrm{E}\left[h(L,R)\right] &= \left(\sum_l P_L(l)\, 0.001l\right) \underbrace{\sum_r P_R(r)}_{=1} + \underbrace{\sum_l P_L(l)}_{=1} \left(\sum_r P_R(r)\, 0.1r\right) \\
&= \sum_l P_L(l)\, 0.001l + \sum_r P_R(r)\, 0.1r \\
&= 0.001\,\mathrm{E}\left[L\right] + 0.1\,\mathrm{E}\left[R\right] \\
&= h(\mathrm{E}\left[L\right], \mathrm{E}\left[R\right]). \qquad (3)
\end{aligned}$$

Thus, we see we have answered the second question first in showing that $\mathrm{E}[h(L,R)] = h(\mathrm{E}[L], \mathrm{E}[R])$. To finish the quiz, we recall from Example 5.15 that $\mathrm{E}[R] = \sum_r r P_R(r) = 8$ Mb/s and $\mathrm{E}[L] = \sum_l l P_L(l) = 1475$ kilobytes. This implies

$$\mathrm{E}\left[h(L,R)\right] = 0.001(1475) + 0.1(8) = 2.275 \text{ cents.} \qquad (4)$$

We note that writing out the above double summations may or may not be instructive but is definitely time consuming. Instead, we can use Theorem 5.10 to bypass those steps by writing

$$\begin{aligned}
\mathrm{E}\left[h(L,R)\right] &= \mathrm{E}\left[0.001L + 0.1R\right] \\
&= 0.001\,\mathrm{E}\left[L\right] + 0.1\,\mathrm{E}\left[R\right] = h(\mathrm{E}\left[L\right], \mathrm{E}\left[R\right]). \qquad (5)
\end{aligned}$$

Quiz 5.8(A) Solution

It is helpful to first make a table that includes the marginal PMFs.

$P_{L,T}(l,t)$	$t=40$	$t=60$	$P_L(l)$
$l=1$	0.15	0.1	0.25
$l=2$	0.3	0.2	0.5
$l=3$	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	

(a) The expected value of L is

$$\mathrm{E}[L] = 1(0.25) + 2(0.5) + 3(0.25) = 2. \tag{1}$$

Since the second moment of L is

$$\mathrm{E}\left[L^2\right] = 1^2(0.25) + 2^2(0.5) + 3^2(0.25) = 4.5, \tag{2}$$

the variance of L is

$$\mathrm{Var}[L] = \mathrm{E}\left[L^2\right] - (\mathrm{E}[L])^2 = 0.5. \tag{3}$$

(b) The expected value of T is

$$\mathrm{E}[T] = 40(0.6) + 60(0.4) = 48. \tag{4}$$

The second moment of T is

$$\mathrm{E}\left[T^2\right] = 40^2(0.6) + 60^2(0.4) = 2400. \tag{5}$$

Thus

$$\mathrm{Var}[T] = \mathrm{E}\left[T^2\right] - (\mathrm{E}[T])^2 = 96. \tag{6}$$

(c) First we need to find

$$\begin{aligned}
\mathrm{E}[LT] &= \sum_{t=40,60} \sum_{l=1}^{3} lt P_{LT}(lt) \\
&= 1(40)(0.15) + 2(40)(0.3) + 3(40)(0.15) \\
&\qquad + 1(60)(0.1) + 2(60)(0.2) + 3(60)(0.1) \\
&= 96.
\end{aligned} \tag{7}$$

The covariance of L and T is

$$\mathrm{Cov}[L,T] = \mathrm{E}[LT] - \mathrm{E}[L]\,\mathrm{E}[T] = 96 - 2(48) = 0. \tag{8}$$

(d) Since $\mathrm{Cov}[L,T] = 0$, the correlation coefficient is $\rho_{L,T} = 0$.

Quiz 5.8(B) Solution

As in the discrete case, the calculations become easier if we first calculate the marginal PDFs $f_X(x)$ and $f_Y(y)$. For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^2 xy\,dy = \frac{1}{2}xy^2\bigg|_{y=0}^{y=2} = 2x. \tag{1}$$

Similarly, for $0 \leq y \leq 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^2 xy\,dx = \frac{1}{2}x^2y\bigg|_{x=0}^{x=1} = \frac{y}{2}. \tag{2}$$

The complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} y/2 & 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

From the marginal PDFs, it is straightforward to calculate the various expectations.

(a) The first and second moments of X are

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\ dx = \int_0^1 2x^2\,dx = \frac{2}{3}. \tag{4}$$

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\ dx = \int_0^1 2x^3\,dx = \frac{1}{2}. \tag{5}$$

The variance of X is

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \frac{1}{18}.$$

(b) The first and second moments of Y are

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\ dy = \int_0^2 \frac{1}{2}y^2\,dy = \frac{4}{3}, \tag{6}$$

$$\mathrm{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\ dy = \int_0^2 \frac{1}{2}y^3\,dy = 2. \tag{7}$$

The variance of Y is

$$\operatorname{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}\left[Y\right])^2 = 2 - \frac{16}{9} = \frac{2}{9}. \tag{8}$$

(c) We start by finding

$$\begin{aligned}
\mathrm{E}\left[XY\right] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\ dx, dy \\
&= \int_0^1 \int_0^2 x^2 y^2\, dx, dy = \left.\frac{x^3}{3}\right|_0^1 \left.\frac{y^3}{3}\right|_0^2 = \frac{8}{9}.
\end{aligned} \tag{9}$$

The covariance of X and Y is then

$$\operatorname{Cov}\left[X,Y\right] = \mathrm{E}\left[XY\right] - \mathrm{E}\left[X\right]\mathrm{E}\left[Y\right] = \frac{8}{9} - \frac{2}{3} \cdot \frac{4}{3} = 0. \tag{10}$$

(d) Since $\operatorname{Cov}[X,Y] = 0$, the correlation coefficient is $\rho_{X,Y} = 0$.

Quiz 5.9 Solution

This problem just requires identifying the various parameters in Definition 5.10. Specifically, from the problem statement, we know $\rho = 1/2$ and

$$\begin{aligned}
\mu_X &= 0, & \mu_Y &= 0, \\
\sigma_X &= 1, & \sigma_Y &= 1.
\end{aligned}$$

Applying these facts to Definition 5.10, we have

$$f_{X,Y}(x,y) = \frac{e^{-2(x^2 - xy + y^2)/3}}{\sqrt{3\pi^2}}. \tag{1}$$

Quiz 5.10 Solution

We find P[C] by integrating the joint PDF over the region of interest. Specifically,

$$\begin{aligned}
\mathrm{P}[C] &= \int_0^{\frac{1}{2}} dy_2 \int_0^{y_2} dy_1 \int_0^{\frac{1}{2}} dy_4 \int_0^{y_4} 4 dy_3 \\
&= 4 \left(\int_0^{\frac{1}{2}} y_2 \, dy_2 \right) \left(\int_0^{\frac{1}{2}} y_4 \, dy_4 \right) \\
&= 4 \left(\frac{1}{2} y_2^2 \Big|_0^{\frac{1}{2}} \right) \left(\frac{1}{2} y_4^2 \Big|_0^{\frac{1}{2}} \right) = 4 \left(\frac{1}{8} \right)^2 = \frac{1}{16}.
\end{aligned} \tag{1}$$

Quiz 6.2 Solution

Since $Y = \sqrt{X}$, the fact that X is nonegative implies Y is non-negative. This implies $F_Y(y) = 0$ for $y < 0$. For $y \geq 0$, we find

$$\begin{aligned} F_Y(y) &= \mathrm{P}\left[\sqrt{X} \leq y\right] \\ &= \mathrm{P}\left[X \leq y^2\right] = F_X\left(y^2\right). \end{aligned} \tag{1}$$

For $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x}$. Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By taking the derivative with respect to y, it follows that the PDF of Y is

$$f_Y(y) = \begin{cases} 2\lambda y e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

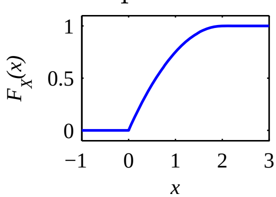
In comparing this result to the Rayleigh PDF given in Appendix A, we observe that Y is a Rayleigh (a) random variable with $a = \sqrt{2\lambda}$.

Quiz 6.3 Solution

(a) Since X is always nonnegative, $F_X(x) = 0$ for $x < 0$. Also, $F_X(x) = 1$ for $x \geq 2$ since its always true that $x \leq 2$. Lastly, for $0 \leq x \leq 2$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy = \int_0^x (1 - y/2)\, dy = x - x^2/4. \tag{1}$$

The complete CDF of X is



$$F_X(x) = \begin{cases} 0 & x < 0, \\ x - x^2/4 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases} \tag{2}$$

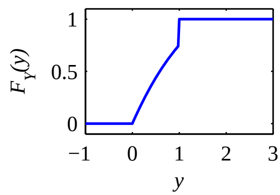
(b) The probability that $Y = 1$ is

$$\begin{aligned} \mathrm{P}[Y=1] &= \mathrm{P}[X \geq 1] \\ &= 1 - F_X(1) = 1/4. \end{aligned} \tag{3}$$

(c) Since X is nonnegative, Y is also nonnegative. Thus $F_Y(y) = 0$ for $y < 0$. Also, because $Y \leq 1$, $F_Y(y) = 1$ for all $y \geq 1$. Finally, for $0 < y < 1$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}[Y \leq y] \\ &= \mathrm{P}[X \leq y] = F_X(y). \end{aligned} \tag{4}$$

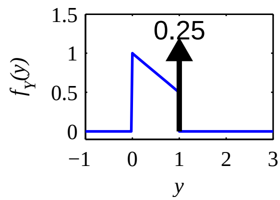
Using the CDF $F_X(x)$, the complete expression for the CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y - y^2/4 & 0 \leq y < 1, \\ 1 & y \geq 1. \end{cases} \tag{5}$$

As expected, we see that the jump in $F_Y(y)$ at $y = 1$ is exactly equal to $\mathrm{P}[Y = 1]$.

(d) By taking the derivative of $F_Y(y)$, we obtain the PDF $f_Y(y)$. Note that when $y < 0$ or $y > 1$, the PDF is zero.



$$f_Y(y) = \begin{cases} 1 - \frac{y}{2} + \frac{\delta(y-1)}{4} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Quiz 6.4(A) Solution

The time required for the transfer is $T = 8L/B$. For each pair of values of L and B, we can calculate the time T needed for the transfer. We can write these down on the table for the joint PMF of L and B as follows:

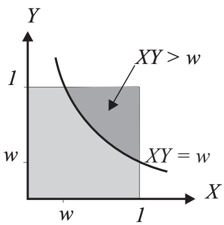
$P_{L,B}(l,b)$	b=512	b=1024	b=2048
$l = 256$	0.20 (T=4)	0.10 (T=2)	0.05 (T=1)
$l = 768$	0.05 (T=12)	0.10 (T=6)	0.20 (T=3)
$l = 1536$	0.00 (T=24)	0.10 (T=12)	0.20 (T=6)

From the table, writing down the PMF of T is just bookkeeping. For example $P[T = 6] = 0.1 + 0.2 = 0.3$. The complete table of the PMF is

t	1	2	3	4	6	12
$P_T(t)$	0.05	0.1	0.2	0.2	0.3	0.15

Quiz 6.4(B) Solution

First, we observe that since $0 \le X \le 1$ and $0 \le Y \le 1$, $W = XY$ satisfies $0 \le W \le 1$. Thus $f_W(0) = 0$ and $f_W(1) = 1$.



For $0 < w < 1$, we calculate the CDF $F_W(w) = P[W \le w]$. As we see in the figure, the calculus is simpler if we integrate over the region $XY > w$. The calculus is

$$\begin{aligned} F_W(w) &= 1 - P[XY > w] \\ &= 1 - \int_w^1 \int_{w/x}^1 dy\,dx \\ &= 1 - \int_w^1 (1 - w/x)\,dx \\ &= \left(x - w \ln x|_{x=w}^{x=1}\right) \\ &= w - w \ln w. \end{aligned} \tag{1}$$

For $0 \le w \le 1$, the PDF is

$$f_W(w) = \frac{dF_W(w)}{dw} = -\ln w. \tag{2}$$

The complete PDF of W is

$$f_W(w) = \begin{cases} 0 & w < 0, \\ -\ln w & 0 \le w \le 1, \\ 0 & w > 1. \end{cases} \tag{3}$$

Quiz 6.5 Solution

Random variables X and Y have PDFs

$$f_X(x) = \begin{cases} 3e^{-3x} & x \ge 0, \\ 0 & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 2e^{-2y} & y \ge 0, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

Since X and Y are nonnegative, $W = X + Y$ is nonnegative and $f_W(w) = 0$ for $w < 0$. For $w > 0$, we use Theorem 6.9 to write

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(w-y)\, f_Y(y)\, dy \\ &= 6\int_0^w e^{-3(w-y)} e^{-2y}\, dy = 6e^{-3w} \int_0^w e^y\, dy = 6e^{-3w}\left(e^w - 1\right). \end{aligned} \tag{2}$$

The complete PDF of W is

$$f_W(w) = \begin{cases} 6(e^{-2w} - e^{-3w}) & w \ge 0, \\ 0 & \text{otherwise}. \end{cases}$$

Quiz 6.6 Solution

Your printing of this textbook may have typo. The PDF of V should be

$$f_V(v) = \begin{cases} (v+5)/72 & -5 \le v \le 7, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

First we find the corresponding CDF $F_V(v)$. For $-5 \le v \le 7$,

$$F_V(v) = \int_{-\infty}^{v} f_V(u)\, du = \int_{-5}^{v} \frac{u+5}{72}\, du \tag{2}$$

$$= \left. \frac{(u+5)^2}{144} \right|_{-5}^{v} = \frac{(v+5)^2}{144}. \tag{3}$$

The complete CDF of V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144 & -5 \le v \le 7, \\ 1 & v > 7. \end{cases} \tag{4}$$

Now that we found the CDF $F_V(v)$, we can use Theorem 6.5. Over the interval $-5 \le v \le 7$, we find the inverse of the CDF by solving

$$u = F_V(v) = \frac{(v+5)^2}{144} \tag{5}$$

for v as a function of u. This yields $v = 12\sqrt{u} - 5$. Thus, when U is a uniform $(0, 1)$ random variable, the function

$$V = 12\sqrt{U} - 5 \tag{6}$$

generates samples of random variable V. In terms of MATLAB, the code is simple:

```
function V = Vsample(m)
V=12*sqrt(rand(1,m))-5;
```

In `Vsample.m`, m samples of a uniform $(0, 1)$ random variable are given by `rand(1,m)`. Here is a sample output

```
>> V=Vsample(5)
V =
    6.7402    3.3603    5.7350   -0.4799    2.7932
```

Quiz 7.1(A) Solution

(a) From the problem statement, we learn that the conditional PMF of N given the event I is

$$P_{N|I}(n) = \begin{cases} 0.02 & n = 1, 2, \ldots, 50, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Also from the problem statement, the conditional PMF of N given the event T is

$$P_{N|T}(n) = \begin{cases} 0.2 & n = 1, \ldots, 5, \\ 0 & \text{otherwise.} \end{cases}$$

(c) The problem statement tells us that $\mathrm{P}[T] = 1 - \mathrm{P}[I] = 3/4$. From Theorem 7.2, we find the PMF of N is

$$P_N(n) = P_{N|T}(n)\,\mathrm{P}[T] + P_{N|I}(n)\,\mathrm{P}[I] = \begin{cases} 0.155 & n = 1, \ldots, 5, \\ 0.005 & n = 6, \ldots, 50, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(d) First we find

$$\mathrm{P}[N \le 10] = \sum_{n=1}^{10} P_N(n) = (0.155)(5) + (0.005)(5) = 0.80. \tag{2}$$

By Theorem 7.1, the conditional PMF of N given $N \le 10$ is

$$\begin{aligned} P_{N|N\le 10}(n) &= \begin{cases} \frac{P_N(n)}{\mathrm{P}[N\le 10]} & n \le 10, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{0.155}{0.8} = 0.19375 & n = 1, \ldots, 5, \\ \frac{0.005}{0.8} = 0.00625 & n = 6, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

Quiz 7.1(B) Solution

From the problem statement,

$$f_Y(y) = \begin{cases} 1/10 & 0 < y < 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Using this PDF and Definition 7.3, the parts are straightforward.

(a) $\mathrm{P}[Y \leq 6] = \int_{-\infty}^{6} f_Y(y)\,dy = \int_0^6 (1/10)\,dy = 0.6\,.$

(b) From Definition 7.3, the conditional PDF of Y given $Y \leq 6$ is

$$f_{Y|Y\leq 6}(y) = \begin{cases} \frac{f_Y(y)}{\mathrm{P}[Y\leq 6]} & y \leq 6, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1/6 & 0 \leq y \leq 6, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) The probability $Y > 8$ is

$$\mathrm{P}[Y > 8] = \int_8^{10} \frac{1}{10}\,dy = 0.2. \tag{3}$$

(d) From Definition 7.3, the conditional PDF of Y given $Y > 8$ is

$$f_{Y|Y>8}(y) = \begin{cases} \frac{f_Y(y)}{\mathrm{P}[Y>8]} & y > 8, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \frac{1}{2} & 8 < y \leq 10, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Quiz 7.2(A) Solution

We refer to the solution of Quiz 7.1(A) for $P_{N|N\leq 10}(n)$.

(a) Given $P_{N|N\leq 10}(n)$, calculating a conditional expected value is the same as for any other expected value except we use the conditional PMF.

$$\begin{aligned} \mathrm{E}[N|N \leq 10] &= \sum_n n P_{N|N\leq 10}(n) \\ &= \sum_{n=1}^{5} 0.19375n + \sum_{n=6}^{10} 0.00625n = 3.15625. \end{aligned} \tag{1}$$

(b) For the conditional variance, we first find the conditional second moment

$$
\begin{aligned}
\mathrm{E}\left[N^2|N \le 10\right] &= \sum_n n^2 P_{N|N\le 10}(n) \\
&= \sum_{n=1}^{5} 0.19375 n^2 + \sum_{n=6}^{10} 0.00625 n^2 \\
&= 0.19375(55) + 0.00625(330) = 12.719. \qquad (2)
\end{aligned}
$$

The conditional variance is

$$
\begin{aligned}
\mathrm{Var}[N|N \le 10] &= \mathrm{E}\left[N^2|N \le 10\right] - \left(\mathrm{E}\left[N|N \le 10\right]\right)^2 \\
&= 12.719 - (3.156)^2 = 2.757. \qquad (3)
\end{aligned}
$$

Quiz 7.2(B) Solution

We refer to the solution of Quiz 7.1(B) for the conditional PDFs $f_{Y|Y\le 6}(y)$ and $f_{Y|Y>8}(y)$.

(a) From $f_{Y|Y\le 6}(y)$, the conditional expectation is

$$
\mathrm{E}\left[Y|Y \le 6\right] = \int_{-\infty}^{\infty} y f_{Y|Y\le 6}(y)\, dy = \int_0^6 \frac{y}{6}\, dy = 3. \qquad (1)
$$

(b) From the conditional PDF $f_{Y|Y>8}(y)$, we see that given $Y > 8$, Y is conditionally a continuous uniform $(a = 8, b = 10)$ random variable. Thus,

$$
\mathrm{Var}[Y|Y > 8] = (b-a)^2/12 = 1/3. \qquad (2)
$$

Quiz 7.3(A) Solution

Since the event $V > 80$ occurs only for the pairs $(L, X) = (2, 60)$, $(L, X) = (3, 40)$ and $(L, X) = (3, 60)$,

$$\mathrm{P}[A] = \mathrm{P}[V > 80] = P_{L,X}(2, 60) + P_{L,X}(3, 40) + P_{L,X}(3, 60) = 0.45. \tag{1}$$

By Definition 7.6,

$$P_{L,X|A}(l, X) = \begin{cases} \frac{P_{L,X}(l,x)}{\mathrm{P}[A]} & lx > 80, \\ 0 & \text{otherwise.} \end{cases}$$

We can represent this conditional PMF in the following table:

$P_{L,X\|A}(l, x)$	$x = 40$	$x = 60$
$l = 1$	0	0
$l = 2$	0	4/9
$l = 3$	1/3	2/9

The conditional expectation of V can be found from the conditional PMF.

$$\mathrm{E}[V|A] = \sum_l \sum_x lxP_{L,X|A}(l, x) = (120)\frac{4}{9} + (120)\frac{1}{3} + (180)\frac{2}{9} = 133\ \frac{1}{3}. \tag{2}$$

For the conditional variance $\mathrm{Var}[V|A]$, we first find the conditional second moment

$$\begin{aligned} \mathrm{E}\left[V^2|A\right] &= \sum_l \sum_x (lx)^2 P_{L,X|A}(l, x) \\ &= (120)^2\frac{4}{9} + (120)^2\frac{1}{3} + (180)^2\frac{2}{9} = 18,400. \end{aligned} \tag{3}$$

It follows that

$$\mathrm{Var}[V|A] = \mathrm{E}\left[V^2|A\right] - (\mathrm{E}[V|A])^2 = 622\ \frac{2}{9} \tag{4}$$

Quiz 7.3(B) Solution

For continuous random variables X and Y, we first calculate the probability of the conditioning event.

$$\mathrm{P}[B] = \iint_B f_{X,Y}(x,y)\,dx\,dy = \int_{40}^{60}\int_{80/y}^{3} \frac{xy}{4000}\,dx\,dy. \tag{1}$$

A little calculus yields

$$\begin{aligned}\mathrm{P}[B] &= \int_{40}^{60} \frac{y}{4000}\left(\left.\frac{x^2}{2}\right|_{80/y}^{3}\right)dy\\ &= \int_{40}^{60}\frac{y}{4000}\left(\frac{9}{2}-\frac{3200}{y^2}\right)dy = \frac{9}{8}-\frac{4}{5}\ln\frac{3}{2}.\end{aligned} \tag{2}$$

In fact, $\mathrm{P}[B] \approx 0.801$. The conditional PDF of X and Y is

$$f_{X,Y|B}(x,y) = \begin{cases}\frac{f_{X,Y}(x,y)}{\mathrm{P}[B]} & (x,y)\in B,\\ 0 & \text{otherwise,}\end{cases} = \begin{cases} Kxy & \substack{40\le y\le 60,\\ 80/y\le x\le 3,}\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

where $K = (4000\,\mathrm{P}[B])^{-1}$. The conditional expectation of W given event B is

$$\mathrm{E}[W|B] = \iint xy f_{X,Y|B}(x,y)\,dx\,dy = \int_{40}^{60}\int_{80/y}^{3} Kx^2y^2\,dx\,dy. \tag{4}$$

These next steps are just calculus:

$$\begin{aligned}\mathrm{E}[W|B] &= \frac{K}{3}\int_{40}^{60} y^2x^3\Big|_{x=80/y}^{x=3}\,dy\\ &= \frac{K}{3}\int_{40}^{60}\left(27y^2-80^3/y\right)dy = \frac{K}{3}\left(9y^3-80^3\ln y\right)\Big|_{40}^{60} \approx 120.78.\end{aligned} \tag{5}$$

The conditional second moment of K given B is

$$\mathrm{E}\left[W^2|B\right] = \iint (xy)^2 f_{X,Y|B}(x,y)\,dx\,dy = \int_{40}^{60}\int_{80/y}^{3} Kx^3y^3\,dx\,dy. \tag{6}$$

With a final bit of calculus,

$$\begin{aligned}
\mathrm{E}\left[W^2|B\right] &= \frac{K}{4}\int_{40}^{60} y^3x^4\Big|_{x=80/y}^{x=3}\,dy \\
&= \frac{K}{4}\int_{40}^{60}\left(81y^3 - 80^4/y\right)\,dy = \frac{K}{4}\left(\frac{81}{4}y^4 - 80^4\ln y\right)\Bigg|_{40}^{60} \\
&\approx 16{,}116.10. \qquad (7)
\end{aligned}$$

It follows that $\mathrm{Var}[W|B] = \mathrm{E}[W^2|B] - (\mathrm{E}[W|B])^2 \approx 1528.30$.

Quiz 7.4(A) Solution

(a) The joint PMF of X and Y can be found from the marginal and conditional PMFs via $P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x)$. Incorporating the information from the given conditional PMFs can be confusing, however. Consequently, we note that X has range $S_X = \{0,2\}$ and Y has range $S_Y = \{0,1\}$. A table of the joint PMF will include all four possible combinations of X and Y. The general form of the table is

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	$P_{Y\|X}(0\|0)P_X(0)$	$P_{Y\|X}(1\|0)P_X(0)$
$x=2$	$P_{Y\|X}(0\|2)P_X(2)$	$P_{Y\|X}(1\|2)P_X(2)$

Substituting values from $P_{Y|X}(y|x)$ and $P_X(x)$, we have

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	(0.8)(0.4)	(0.2)(0.4)
$x=2$	(0.5)(0.6)	(0.5)(0.6)

,

which simplifies to

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	0.32	0.08
$x=2$	0.3	0.3

.

(b) From the joint PMF $P_{X,Y}(x,y)$, we can calculate $P_Y(0) = 0.32 + 0.3 = 0.62$ and the conditional PMF

$$P_{X|Y}(x|0) = \frac{P_{X,Y}(x,0)}{P_Y(0)} = \begin{cases} \frac{0.32}{0.62} = \frac{16}{31} & x = 0, \\ \frac{0.3}{0.62} = \frac{15}{31} & x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Quiz 7.4(B) Solution

(a) The joint PDF of X and Y is

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x) = \begin{cases} 6y & 0 \le y \le x, 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) To find $f_{X|Y}(x|1/2)$, we first find

$$f_Y(1/2) = \int_{-\infty}^{\infty} f_{X,Y}(x, 1/2)\, dx.$$

For this integral, we keep in mind that $f_{X,Y}(x,y)$ is nonzero for $y \le x \le 1$, Specifically, for $y = 1/2$, we integrate over $1/2 \le x \le 1$:

$$f_Y(1/2) = \int_{1/2}^{1} 6(1/2)\, dx = 3/2. \tag{1}$$

For $1/2 \le x \le 1$, the conditional PDF of X given $Y = 1/2$ is

$$f_{X|Y}(x|1/2) = \frac{f_{X,Y}(x,1/2)}{f_Y(1/2)} = \frac{6(1/2)}{3/2} = 2. \tag{2}$$

For $x < 1/2$ or $x > 1$, $f_{X|Y}(x|1/2) = 0$. Thus given $Y = 1/2$, the X has the continuous uniform $(1/2, 1)$ PDF

$$f_{X|Y}(x|1/2) = \begin{cases} 2 & \frac{1}{2} \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Quiz 7.5(A) Solution

(a) Given the conditional PMF $P_{Y|X}(y|2)$, it is easy to calculate the conditional expectation

$$\mathrm{E}[Y|X=2] = \sum_{y=0}^{1} y P_{Y|X}(y|2) = (0)(0.5) + (1)(0.5) = 0.5. \tag{1}$$

(b) We can calculate the conditional variance $\mathrm{Var}[X|Y=0]$ using the conditional PMF $P_{X|Y}(x|0)$. First we calculate the conditional expected value

$$\mathrm{E}[X|Y=0] = \sum_x x P_{X|Y}(x|0) = 0 \cdot \frac{16}{31} + 2 \cdot \frac{15}{31} = \frac{30}{31}. \tag{2}$$

The conditional second moment is

$$\mathrm{E}\left[X^2|Y=0\right] = \sum_x x^2 P_{X|Y}(x|0) = 0^2 \frac{16}{31} + 2^2 \frac{15}{31} = \frac{60}{31}. \tag{3}$$

The conditional variance is then

$$\mathrm{Var}[X|Y=0] = \mathrm{E}\left[X^2|Y=0\right] - (\mathrm{E}[X|Y=0])^2 = 960/961. \tag{4}$$

Quiz 7.5(B) Solution

(a) From the conditional PDF $f_{Y|X}(y|x)$ given in Quiz 7.4(B),

$$f_{Y|X}(y|1/2) = \begin{cases} 8y & 0 \le y \le 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Now we calculate the conditional expected value

$$\mathrm{E}[Y|X=1/2] = \int_0^{1/2} y(8y)\,dy = 8y^3/3\Big|_0^{1/2} = 1/3. \tag{2}$$

(b) From the solution to Quiz 7.4(B), we see that given $Y = 1/2$, the conditional PDF of X is uniform $(1/2, 1)$. Thus, by the definition of the uniform (a, b) PDF,

$$\text{Var}[X|Y = 1/2] = \frac{(1 - 1/2)^2}{12} = \frac{1}{48}.$$

Quiz 7.6 Solution

Since X and Y are bivariate Gaussian random variables with $\rho = 1/2$, $\mu_X = \mu_Y = 0$, and $\sigma_X = \sigma_Y = 1$, Theorem 7.16 tells us that given $Y = y$, X is conditionally Gaussian with parameters

$$\tilde{\mu}_X(y) = \rho y = \frac{y}{2}, \quad \tilde{\sigma}_X^2 = 1 - \rho^2. \tag{1}$$

For $y = 2$, we have

$$\tilde{\mu}_X = \tilde{\mu}_X(2) = 1 \qquad\qquad \tilde{\sigma}_X^2 = 3/4. \tag{2}$$

The conditional PDF of X is

$$f_{X|Y}(x|2) = \frac{1}{\sqrt{2\pi\tilde{\sigma}_X^2}} e^{-(x-\tilde{\mu}_X)^2/2\tilde{\sigma}_X^2} = \frac{1}{\sqrt{3\pi/2}} e^{-2(x-1)^2/3}. \tag{3}$$

Quiz 7.7 Solution

One straightforward method is to follow the approach of Example 5.27. Instead, we use an alternate approach. First we observe that X has the discrete uniform $(1, 4)$ PMF. Also, given $X = x$, Y has a discrete uniform $(1, x)$ PMF. That is,

$$P_X(x) = \begin{cases} 1/4 & x = 1, 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases} \qquad P_{Y|X}(y|x) = \begin{cases} 1/x & y = 1, \ldots, x, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Given $X = x$, and an independent uniform $(0, 1)$ random variable U, we can generate a sample value of Y with a discrete uniform $(1, x)$ PMF via $Y = \lceil xU \rceil$. This observation prompts the following program:

```
function xy=dtrianglerv(m)
sx=[1;2;3;4];
px=0.25*ones(4,1);
x=finiterv(sx,px,m);
y=ceil(x.*rand(m,1));
xy=[x';y'];
```

Quiz 8.1 Solution

By definition of $\mathbf{A}$, $Y_1 = X_1$, $Y_2 = X_2 - X_1$ and $Y_3 = X_3 - X_2$. Since $0 < X_1 < X_2 < X_3$, each Y_i must be a strictly positive integer. Thus, for $y_1, y_2, y_3 \in \{1, 2, \ldots\}$,

$$
\begin{aligned}
P_{\mathbf{Y}}(\mathbf{y}) &= \mathrm{P}\left[Y_1 = y_1, Y_2 = y_2, Y_3 = y_3\right] \\
&= \mathrm{P}\begin{bmatrix} X_1 = & y_1, \\ X_2 - X_1 = & y_2, \\ X_3 - X_2 = & y_3 \end{bmatrix} \\
&= \mathrm{P}\begin{bmatrix} X_1 = & y_1, \\ X_2 = & y_2 + y_1, \\ X_3 = & y_3 + y_2 + y_1 \end{bmatrix} \\
&= P_{\mathbf{X}}(y_1, y_2 + y_1, y_3 + y_2 + y_1) && (1) \\
&= (1-p)^3 p^{y_1+y_2+y_3}. && (2)
\end{aligned}
$$

With $\mathbf{a} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$ and $q = 1 - p$, the joint PMF of $\mathbf{Y}$ is

$$
P_{\mathbf{Y}}(\mathbf{y}) = \begin{cases} qp^{\mathbf{a}'\mathbf{y}} & y_1, y_2, y_3 \in \{1, 2, \ldots\}, \\ 0 & \text{otherwise.} \end{cases}
$$

Quiz 8.2 Solution

In the PDF $f_{\mathbf{Y}}(\mathbf{y})$, the components have dependencies as a result of the ordering constraints $Y_1 \leq Y_2$ and $Y_3 \leq Y_4$. We can separate these constraints by creating the vectors

$$
\mathbf{V} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} Y_3 \\ Y_4 \end{bmatrix}. \tag{1}
$$

The joint PDF of $\mathbf{V}$ and $\mathbf{W}$ is

$$
f_{\mathbf{V},\mathbf{W}}(\mathbf{v}, \mathbf{w}) = \begin{cases} 4 & \begin{matrix} 0 \leq v_1 \leq v_2 \leq 1; \\ 0 \leq w_1 \leq w_2 \leq 1, \end{matrix} \\ 0 & \text{otherwise.} \end{cases} \tag{2}
$$

We must verify that $\mathbf{V}$ and $\mathbf{W}$ are independent. For $0 \leq v_1 \leq v_2 \leq 1$,

$$
\begin{aligned}
f_{\mathbf{V}}(\mathbf{v}) &= \iint f_{\mathbf{V},\mathbf{W}}(\mathbf{v},\mathbf{w})\, dw_1\, dw_2 \\
&= \int_0^1 \left(\int_{w_1}^1 4\, dw_2 \right) dw_1 \\
&= \int_0^1 4(1-w_1)\, dw_1 = 2.
\end{aligned} \tag{3}
$$

Similarly, for $0 \leq w_1 \leq w_2 \leq 1$,

$$
\begin{aligned}
f_{\mathbf{W}}(\mathbf{w}) &= \iint f_{\mathbf{V},\mathbf{W}}(\mathbf{v},\mathbf{w})\, dv_1\, dv_2 \\
&= \int_0^1 \left(\int_{v_1}^1 4\, dv_2 \right) dv_1 = 2.
\end{aligned} \tag{4}
$$

It follows that $\mathbf{V}$ and $\mathbf{W}$ have PDFs

$$
f_{\mathbf{V}}(\mathbf{v}) = \begin{cases} 2 & 0 \leq v_1 \leq v_2 \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}
$$

$$
f_{\mathbf{W}}(\mathbf{w}) = \begin{cases} 2 & 0 \leq w_1 \leq w_2 \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}
$$

It is easy to verify that $f_{\mathbf{V},\mathbf{W}}(\mathbf{v},\mathbf{w}) = f_{\mathbf{V}}(\mathbf{v}) f_{\mathbf{W}}(\mathbf{w})$, confirming that $\mathbf{V}$ and $\mathbf{W}$ are independent vectors.

Quiz 8.3(A) Solution

Referring to Theorem 2.9, each test is a subexperiment with three possible outcomes: L, A and R. In five trials, the vector $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$ indicating the number of outcomes of each subexperiment has the multinomial PMF

$$
P_{\mathbf{X}}(\mathbf{x}) = \binom{5}{x_1, x_2, x_3} 0.3^{x_1} 0.6^{x_2} 0.1^{x_3}.
$$

We can find the marginal PMF for each X_i from the joint PMF $P_{\mathbf{X}}(\mathbf{x})$; however it is simpler to just start from first principles and observe that X_1 is the number of occurrences of L in five independent tests. If we view each test as a trial with success probability $\mathrm{P}[L] = 0.3$, we see that X_1 is a binomial $(n, p) = (5, 0.3)$ random variable. Similarly, X_2 is a binomial $(5, 0.6)$ random variable and X_3 is a binomial $(5, 0.1)$ random variable. That is, for $p_1 = 0.3$, $p_2 = 0.6$ and $p_3 = 0.1$,

$$P_{X_i}(x) = \binom{5}{x} p_i^x (1 - p_i)^{5-x}. \tag{1}$$

From the marginal PMFs, we see that X_1, X_2 and X_3 are not independent. Hence, we must use Theorem 8.1 to find the PMF of W. In particular, since $X_1 + X_2 + X_3 = 5$ and since each X_i is non-negative, $P_W(0) = P_W(1) = 0$. Furthermore,

$$\begin{aligned} P_W(2) &= P_{\mathbf{X}}(1, 2, 2) + P_{\mathbf{X}}(2, 1, 2) + P_{\mathbf{X}}(2, 2, 1) \\ &= \frac{5! 0.3(0.6)^2(0.1)^2}{2!2!1!} + \frac{5! 0.3^2(0.6)(0.1)^2}{2!2!1!} + \frac{5! 0.3^2(0.6)^2(0.1)}{2!2!1!} \\ &= 0.1458. \end{aligned} \tag{2}$$

In addition, for $w = 3$, $w = 4$, and $w = 5$, the event $W = w$ occurs if and only if one of the mutually exclusive events $X_1 = w$, $X_2 = w$, or $X_3 = w$ occurs. Thus,

$$P_W(3) = \sum_{i=1}^{3} P_{X_i}(3) = 0.486, \tag{3}$$

$$P_W(4) = \sum_{i=1}^{3} P_{X_i}(4) = 0.288, \tag{4}$$

$$P_W(5) = \sum_{i=1}^{3} P_{X_i}(5) = 0.0802. \tag{5}$$

Quiz 8.3(B) Solution

Since each $Y_i = 2X_i + 4$, we can apply Theorem 8.5 to write

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{2^3} f_{\mathbf{X}}\left(\frac{y_1-4}{2}, \frac{y_2-4}{2}, \frac{y_3-4}{2}\right) \\ &= \begin{cases} (1/8)e^{-(y_3-4)/2} & 4 \le y_1 \le y_2 \le y_3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

Note that for other matrices $\mathbf{A}$, the constraints on $\mathbf{y}$ resulting from the constraints $0 \le X_1 \le X_2 \le X_3$ can be much more complicated.

Quiz 8.4 Solution

To solve this problem, we need to find the expected values $\mathrm{E}[X_i]$ and $\mathrm{E}[X_i X_j]$ for each I and j. To do this, we need the marginal PDFs $f_{X_i}(x_i)$ and $f_{X_i,X_j}(x_i, x_j)$. First we note that each marginal PDF is nonzero only if any subset of the x_i obeys the ordering contraints $0 \le x_1 \le x_2 \le x_3 \le 1$. Within these constraints, we have

$$f_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x})\, dx_3 = \int_{x_2}^{1} 6\, dx_3 = 6(1 - x_2), \tag{1}$$

and

$$f_{X_2,X_3}(x_2, x_3) = \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x})\, dx_1 = \int_{0}^{x_2} 6\, dx_1 = 6x_2, \tag{2}$$

and

$$f_{X_1,X_3}(x_1, x_3) = \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x})\, dx_2 = \int_{x_1}^{x_3} 6\, dx_2 = 6(x_3 - x_1). \tag{3}$$

In particular, we must keep in mind that $f_{X_1,X_2}(x_1, x_2) = 0$ unless $0 \le x_1 \le x_2 \le 1$, $f_{X_2,X_3}(x_2, x_3) = 0$ unless $0 \le x_2 \le x_3 \le 1$, and that $f_{X_1,X_3}(x_1, x_3) = 0$

unless $0 \le x_1 \le x_3 \le 1$. The complete expressions are

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 6(1-x_2) & 0 \le x_1 \le x_2 \le 1, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

$$f_{X_2,X_3}(x_2,x_3) = \begin{cases} 6x_2 & 0 \le x_2 \le x_3 \le 1, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

and

$$f_{X_1,X_3}(x_1,x_3) = \begin{cases} 6(x_3-x_1) & 0 \le x_1 \le x_3 \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

Now we can find the marginal PDFs. When $0 \le x_i \le 1$ for each x_i,

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\; dx_2 \\ &= \int_{x_1}^{1} 6(1-x_2)\,dx_2 = 3(1-x_1)^2. \end{aligned} \tag{7}$$

$$\begin{aligned} f_{X_2}(x_2) &= \int_{-\infty}^{\infty} f_{X_2,X_3}(x_2,x_3)\; dx_3 \\ &= \int_{x_2}^{1} 6x_2\,dx_3 = 6x_2(1-x_2). \end{aligned} \tag{8}$$

$$\begin{aligned} f_{X_3}(x_3) &= \int_{-\infty}^{\infty} f_{X_2,X_3}(x_2,x_3)\; dx_2 \\ &= \int_{0}^{x_3} 6x_2\,dx_2 = 3x_3^2. \end{aligned} \tag{9}$$

The complete expressions are

$$f_{X_1}(x_1) = \begin{cases} 3(1-x_1)^2 & 0 \le x_1 \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

$$f_{X_2}(x_2) = \begin{cases} 6x_2(1-x_2) & 0 \le x_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

$$f_{X_3}(x_3) = \begin{cases} 3x_3^2 & 0 \le x_3 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Now we can find the components $\mathrm{E}[X_i] = \int_{-\infty}^{\infty} x f_{X_i}(x)\,dx$ of $\boldsymbol{\mu}_X$.

$$\mathrm{E}\left[X_1\right] = \int_0^1 3x(1-x)^2\,dx = 1/4, \tag{13}$$

$$\mathrm{E}\left[X_2\right] = \int_0^1 6x^2(1-x)\,dx = 1/2, \tag{14}$$

$$\mathrm{E}\left[X_3\right] = \int_0^1 3x^3\,dx = 3/4. \tag{15}$$

To find the correlation matrix $\mathbf{R}_X$, we need to find $\mathrm{E}[X_iX_j]$ for all i and j. We start with the second moments:

$$\mathrm{E}\left[X_1^2\right] = \int_0^1 3x^2(1-x)^2\,dx = \frac{1}{10}. \tag{16}$$

$$\mathrm{E}\left[X_2^2\right] = \int_0^1 6x^3(1-x)\,dx = \frac{3}{10}. \tag{17}$$

$$\mathrm{E}\left[X_3^2\right] = \int_0^1 3x^4\,dx = \frac{3}{5}. \tag{18}$$

Using marginal PDFs, the cross terms are

$$\begin{aligned}
\mathrm{E}\left[X_1X_2\right] &= \iint x_1x_2 f_{X_1,X_2}(x_1,x_2)\,,\,dx_1\,dx_2 \\
&= \int_0^1 \left(\int_{x_1}^1 6x_1x_2(1-x_2)\,dx_2\right) dx_1 \\
&= \int_0^1 [x_1 - 3x_1^3 + 2x_1^4]\,dx_1 = \frac{3}{20}.
\end{aligned} \tag{19}$$

$$
\begin{aligned}
\mathrm{E}\left[X_2X_3\right] &= \int_0^1 \int_{x_2}^1 6x_2^2 x_3 \, dx_3 \, dx_2 \\
&= \int_0^1 [3x_2^2 - 3x_2^4] \, dx_2 = \frac{2}{5}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left[X_1X_3\right] &= \int_0^1 \int_{x_1}^1 6x_1 x_3 (x_3 - x_1) \, dx_3 \, dx_1 \\
&= \int_0^1 \left((2x_1 x_3^3 - 3x_1^2 x_3^2)\Big|_{x_3=x_1}^{x_3=1} \right) dx_1 \\
&= \int_0^1 [2x_1 - 3x_1^2 + x_1^4] \, dx_1 = 1/5. \qquad (20)
\end{aligned}
$$

Summarizing the results, $\mathbf{X}$ has correlation matrix

$$
\mathbf{R}_X = \begin{bmatrix} 1/10 & 3/20 & 1/5 \\ 3/20 & 3/10 & 2/5 \\ 1/5 & 2/5 & 3/5 \end{bmatrix}. \qquad (21)
$$

Vector $\mathbf{X}$ has covariance matrix

$$
\begin{aligned}
\mathbf{C}_X &= \mathbf{R}_X - \mathrm{E}\left[\mathbf{X}\right] \mathrm{E}\left[\mathbf{X}\right]' \\
&= \begin{bmatrix} \frac{1}{10} & \frac{3}{20} & \frac{1}{5} \\ \frac{3}{20} & \frac{3}{10} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \qquad (22)
\end{aligned}
$$

This problem shows that even for fairly simple joint PDFs, computing the covariance matrix can be time consuming.

Quiz 8.5 Solution

We observe that $\mathbf{X} = \mathbf{A}\mathbf{Z} + \mathbf{b}$ where

$$
\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \qquad (1)
$$

It follows from Theorem 8.13 that $\boldsymbol{\mu}_X = \mathbf{b}$ and that

$$\mathbf{C}_X = \mathbf{A}\mathbf{A}' = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}.$$

Quiz 8.6 Solution

First, we observe that $Y = \mathbf{A}\mathbf{T}$ where $\mathbf{A} = \begin{bmatrix} 1/31 & 1/31 & \cdots & 1/31 \end{bmatrix}'$. Since $\mathbf{T}$ is a Gaussian random vector, Theorem 8.11 tells us that Y is a 1 dimensional Gaussian vector, i.e., just a Gaussian random variable. The expected value of Y is $\mu_Y = \mu_T = 80$. The covariance matrix of Y is 1×1 and is just equal to $\text{Var}[Y]$. Thus, by Theorem 8.11, $\text{Var}[Y] = \mathbf{A}\mathbf{C}_T\mathbf{A}'$.

In `julytemps.m` shown below, the first two lines generate the 31×31 covariance matrix `CT`, or $\mathbf{C}_T$. Next we calculate $\text{Var}[Y]$. The final step is to use the $\Phi(\cdot)$ function to calculate $P[Y < T]$.

```
function p=julytemps(T);
[D1 D2]=ndgrid((1:31),(1:31));
CT=36./(1+abs(D1-D2));
A=ones(31,1)/31.0;
CY=(A')*CT*A;
p=phi((T-80)/sqrt(CY));
```

Here is the output of `julytemps.m`:

```
>> julytemps([70 75 80 85 90])
ans =
 0.0000 0.0221 0.5000 0.9779 1.0000
```

Note that $P[T \leq 70]$ is not actually zero and that $P[T \leq 90]$ is not actually 1.0000. Its just that the MATLAB's short format output, invoked with the command `format short`, rounds off those probabilities. The long format output resembles:

```
>> format long
>> julytemps([70 75])
ans =
  0.000028442631  0.022073830676
>> julytemps([85 90])
ans =
  0.977926169323  0.999971557368
```

The `ndgrid` function is a useful to way calculate many covariance matrices. However, in this problem, $\mathbf{C}_X$ has a special structure; the i, jth element is

$$C_{\mathbf{T}}(i,j) = c_{|i-j|} = \frac{36}{1+|i-j|}. \tag{1}$$

If we write out the elements of the covariance matrix, we see that

$$\mathbf{C}_{\mathbf{T}} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{30} \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_{30} & \cdots & c_1 & c_0 \end{bmatrix}. \tag{2}$$

This covariance matrix is known as a symmetric Toeplitz matrix. Because Toeplitz covariance matrices are quite common, MATLAB has a `toeplitz` function for generating them. The function `julytemps2` use the `toeplitz` to generate the correlation matrix $\mathbf{C}_{\mathbf{T}}$.

```
function p=julytemps2(T);
c=36./(1+abs(0:30));
CT=toeplitz(c);
A=ones(31,1)/31.0;
CY=(A')*CT*A;
p=phi((T-80)/sqrt(CY));
```

Quiz 9.1 Solution

Let $K_1, \ldots, K_n$ denote a sequence of iid random variables each with PMF

$$P_K(k) = \begin{cases} 1/4 & k = 1, \ldots, 4, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We can write $W_n = K_1 + \cdots + K_n$. First, we note that the first two moments of K_i are

$$\mathrm{E}[K_i] = \frac{1+2+3+4}{4} = 2.5, \tag{2}$$

$$\mathrm{E}\left[K_i^2\right] = \frac{1^2+2^2+3^2+4^2}{4} = 7.5. \tag{3}$$

Thus the variance of K_i is

$$\begin{aligned} \mathrm{Var}[K_i] &= \mathrm{E}\left[K_i^2\right] - (\mathrm{E}[K_i])^2 \\ &= 7.5 - (2.5)^2 = 1.25. \end{aligned} \tag{4}$$

Since $\mathrm{E}[K_i] = 2.5$, the expected value of W_n is

$$\mathrm{E}[W_n] = \mathrm{E}[K_1] + \cdots + \mathrm{E}[K_n] = 2.5n. \tag{5}$$

Since the rolls are independent, the random variables $K_1, \ldots, K_n$ are independent. Hence, by Theorem 9.3, the variance of the sum equals the sum of the variances. That is,

$$\mathrm{Var}[W_n] = \mathrm{Var}[K_1] + \cdots + \mathrm{Var}[K_n] = 1.25n. \tag{6}$$

Quiz 9.2 Solution

The MGF of K is

$$\phi_K(s) = \mathrm{E}\left[e^{sK}\right] = \sum_{k=0}^{4} \frac{1}{5} e^{sk} = \frac{1 + e^s + e^{2s} + e^{3s} + e^{4s}}{5}. \tag{1}$$

We find the moments by taking derivatives. The first derivative of $\phi_K(s)$ is

$$\frac{d\phi_K(s)}{ds} = \frac{e^s + 2e^{2s} + 3e^{3s} + 4e^{4s}}{5}. \tag{2}$$

Evaluating the derivative at $s = 0$ yields

$$\mathrm{E}[K] = \left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = \frac{1+2+3+4}{5} = 2. \tag{3}$$

To find higher-order moments, we continue to take derivatives:

$$\begin{aligned} \mathrm{E}\left[K^2\right] &= \left.\frac{d^2\phi_K(s)}{ds^2}\right|_{s=0} \\ &= \left.\frac{e^s + 4e^{2s} + 9e^{3s} + 16e^{4s}}{5}\right|_{s=0} = 6. \end{aligned} \tag{4}$$

$$\begin{aligned} \mathrm{E}\left[K^3\right] &= \left.\frac{d^3\phi_K(s)}{ds^3}\right|_{s=0} \\ &= \left.\frac{e^s + 8e^{2s} + 27e^{3s} + 64e^{4s})}{5}\right|_{s=0} = 20. \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{E}\left[K^4\right] &= \left.\frac{d^4\phi_K(s)}{ds^4}\right|_{s=0} \\ &= \left.\frac{e^s + 16e^{2s} + 81e^{3s} + 256e^{4s}}{5}\right|_{s=0} = 70.8. \end{aligned} \tag{6}$$

Quiz 9.3(A) Solution

Each K_i has MGF

$$\phi_K(s) = \mathrm{E}\left[e^{sK_i}\right] = \frac{e^s + e^{2s} + \cdots + e^{ns}}{n} = \frac{e^s(1 - e^{ns}}{n(1 - e^s)}. \tag{1}$$

Since the sequence of K_i is independent, Theorem 9.6 says the MGF of J is

$$\phi_J(s) = (\phi_K(s))^m = \frac{e^{ms}(1 - e^{ns})^m}{n^m(1 - e^s)^m}. \tag{2}$$

Quiz 9.3(B) Solution

Since the set of $\alpha^j X_j$ are independent Gaussian random variables, Theorem 9.8 says that W is a Gaussian random variable. Thus to find the PDF of W, we need only find the expected value and variance. Since the expectation of the sum equals the sum of the expectations:

$$\mathrm{E}[W] = \alpha\,\mathrm{E}[X_1] + \alpha^2\,\mathrm{E}[X_2] + \cdots + \alpha^n\,\mathrm{E}[X_n] = 0. \tag{1}$$

Since the $\alpha^j X_j$ are independent, the variance of the sum equals the sum of the variances:

$$\begin{aligned}\operatorname{Var}[W] &= \alpha^2 \operatorname{Var}[X_1] + \alpha^4 \operatorname{Var}[X_2] + \cdots + \alpha^{2n} \operatorname{Var}[X_n] \\ &= \alpha^2 + 2(\alpha^2)^2 + \cdots + n(\alpha^2)^n. \end{aligned} \tag{2}$$

Defining $q = \alpha^2$, we can use Math Fact B.6 to write

$$\operatorname{Var}[W] = \frac{\alpha^2 - \alpha^{2n+2}[1 + n(1-\alpha^2)]}{(1-\alpha^2)^2}. \tag{3}$$

With $\mathrm{E}[W] = 0$ and $\sigma_W^2 = \operatorname{Var}[W]$, we can write the PDF of W as

$$f_W(w) = \frac{1}{\sqrt{2\pi\sigma_W^2}} e^{-w^2/2\sigma_W^2}. \tag{4}$$

Quiz 9.4 Solution

(a) From Table 9.1, each X_i has MGF $\phi_X(s) = 1/(1-s)$ and random variable N has MGF

$$\phi_N(s) = \frac{\frac{1}{5}e^s}{1 - \frac{4}{5}e^s}. \tag{1}$$

From Theorem 9.10, R has MGF

$$\phi_R(s) = \phi_N(\ln \phi_X(s)) = \frac{\frac{1}{5}\phi_X(s)}{1 - \frac{4}{5}\phi_X(s)}. \tag{2}$$

Substituting the expression for $\phi_X(s)$ yields

$$\phi_R(s) = \frac{1/5}{1/5 - s}. \tag{3}$$

(b) From Table 9.1, we see that R has the MGF of an exponential (1/5) random variable. The corresponding PDF is

$$f_R(r) = \begin{cases} (1/5)e^{-r/5} & r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

This quiz is an example of the general result that a geometric sum of exponential random variables is an exponential random variable.

Quiz 9.5 Solution

(a) The expected access time is

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{12} \frac{x}{12}\, dx = 6 \text{ ms.} \tag{1}$$

(b) The second moment of the access time is

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^{12} \frac{x^2}{12}\, dx = 48. \tag{2}$$

The variance of the access time is $\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 12$.

(c) Using X_i to denote the access time of block i, we can write

$$A = X_1 + X_2 + \cdots + X_{12} \tag{3}$$

Since the expectation of the sum equals the sum of the expectations,

$$\mathrm{E}[A] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{12}] = 12\,\mathrm{E}[X] = 72 \text{ ms.} \tag{4}$$

(d) Since the X_i are independent,

$$\mathrm{Var}[A] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_{12}] = 12\,\mathrm{Var}[X] = 144. \tag{5}$$

Thus A has standard deviation $\sigma_A = 12$.

(e) To use the central limit theorem, we use Table 4.2 to evaluate

$$\begin{aligned} \mathrm{P}[A \leq 75] &= \mathrm{P}\left[\frac{A - \mathrm{E}[A]}{\sigma_A} \leq \frac{75 - \mathrm{E}[A]}{\sigma_A}\right] \\ &\approx \Phi\left(\frac{75 - 72}{12}\right) = 0.5987. \end{aligned} \tag{6}$$

Then $\mathrm{P}[A > 75] = 1 - \mathrm{P}[A \leq 75] = 0.4013$.

(f) Once again, we use the central limit theorem and Table 4.2 to estimate

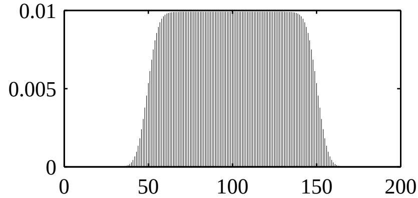
$$\begin{aligned} \mathrm{P}[A < 48] &= \mathrm{P}\left[\frac{A - \mathrm{E}[A]}{\sigma_A} < \frac{48 - \mathrm{E}[A]}{\sigma_A}\right] \\ &\approx \Phi\left(\frac{48 - 72}{12}\right) = 0.0227. \end{aligned} \tag{7}$$

Quiz 9.6 Solution

One solution to this problem is to follow the approach of Example 9.17:

```
%unifbinom100.m
sx=0:100;sy=0:100;
px=binomialpmf(100,0.5,sx);
py=duniformpmf(0,100,sy);
[SX,SY]=ndgrid(sx,sy);
[PX,PY]=ndgrid(px,py);
SW=SX+SY; PW=PX.*PY;
sw=unique(SW);
pw=finitepmf(SW,PW,sw);
pmfplot(sw,pw);
```

Here is a graph of the PMF $P_W(w)$:



With some thought, it should be apparent that the `finitepmf` function is implementing the convolution of the two PMFs.

Quiz 10.1 Solution

An exponential random variable with expected value 1 also has variance 1. By Theorem 10.1, $M_n(X)$ has variance $\text{Var}[M_n(X)] = 1/n$. Hence, we need $n = 100$ samples.

Quiz 10.2 Solution

The train interarrival times X_1, X_2, X_3 are iid exponential (λ) random variables. The arrival time of the third train is

$$W = X_1 + X_2 + X_3. \tag{1}$$

In Theorem 9.9, we found that the sum of three iid exponential (λ) random variables is an Erlang ($n = 3, \lambda$) random variable. From Appendix A, we find that W has expected value and variance

$$\text{E}[W] = 3/\lambda = 6, \tag{2}$$

$$\text{Var}[W] = 3/\lambda^2 = 12. \tag{3}$$

(a) By the Central Limit Theorem,

$$\begin{aligned} \text{P}[W > 20] &= \text{P}\left[\frac{W-6}{\sqrt{12}} > \frac{20-6}{\sqrt{12}}\right] \\ &\approx Q\left(\frac{7}{\sqrt{3}}\right) = 2.66 \times 10^{-5}. \end{aligned}$$

(b) From the Markov inequality, we know that

$$\text{P}[W > 20] \leq \frac{\text{E}[W]}{20} = \frac{6}{20} = 0.3. \tag{4}$$

(c) To use the Chebyshev inequality, we observe that $\text{E}[W] = 6$ and W nonnegative imply

$$\begin{aligned} \text{P}[|W - \text{E}[W]| \geq 14] &= \text{P}[W - 6 \geq 14] + \underbrace{\text{P}[W - 6 \leq -14]}_{=0} \\ &= \text{P}[W \geq 20]. \end{aligned} \tag{5}$$

Thus

$$\begin{aligned}
\mathrm{P}[W \geq 20] &= \mathrm{P}[|W - \mathrm{E}[W]| \geq 14] && (6)\\
&\leq \frac{\mathrm{Var}[W]}{14^2} = \frac{3}{49} = 0.061. && (7)
\end{aligned}$$

(d) For the Chernoff bound, we note that the MGF of W is

$$\phi_W(s) = \left(\frac{\lambda}{\lambda - s}\right)^3 = \frac{1}{(1-2s)^3}. \tag{8}$$

The Chernoff bound states that

$$\mathrm{P}[W > 20] \leq \min_{s \geq 0} e^{-20s}\phi_X(s) = \min_{s \geq 0} \frac{e^{-20s}}{(1-2s)^3}. \tag{9}$$

To minimize $h(s) = e^{-20s}/(1-2s)^3$, we set the derivative of $h(s)$ to zero:

$$\frac{dh(s)}{ds} = \frac{e^{-20s}(40s - 14)}{(1-2s)^4} = 0. \tag{10}$$

This implies $s = 7/20$. Applying $s = 7/20$ into the Chernoff bound yields

$$\mathrm{P}[W > 20] \leq \left.\frac{e^{-20s}}{(1-2s)^3}\right|_{s=\frac{7}{20}} = 0.0338.$$

(e) Theorem 4.11 says that for any $w > 0$, the CDF of the Erlang $(3, \lambda)$ random variable W satisfies

$$F_W(w) = 1 - \sum_{k=0}^{2} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \tag{11}$$

Equivalently, for $\lambda = 1/2$ and $w = 20$,

$$\begin{aligned} \mathrm{P}[W > 20] &= 1 - F_W(20) \\ &= e^{-10}\left(1 + \frac{10}{1!} + \frac{10^2}{2!}\right) \\ &= 61e^{-10} = 0.0028. \end{aligned} \tag{12}$$

Although the Chernoff bound is weak in that it overestimates the probability by a factor of 12, it is a valid bound. By contrast, the Central Limit Theorem approximation grossly underestimates the true probability.

Quiz 10.3 Solution

(a) Since X is a Bernoulli random variable with parameter $p = 0.8$, we can look up in Appendix A to find that $\mathrm{E}[X] = p = 0.8$ and variance

$$\mathrm{Var}[X] = p(1-p) = (0.8)(0.2) = 0.16. \tag{1}$$

(b) By Theorem 10.1,

$$\mathrm{Var}[M_{100}(X)] = \frac{\mathrm{Var}[X]}{100} = 0.0016. \tag{2}$$

(c) Theorem 10.5 uses the Chebyshev inequality to show that the sample mean satisfies

$$\mathrm{P}[|M_n(X) - \mathrm{E}[X]| \geq c] \leq \frac{\mathrm{Var}[X]}{nc^2}. \tag{3}$$

Note that $\mathrm{E}[X] = P_X(1) = p$. To meet the specified requirement, we choose $c = 0.05$ and $n = 100$. Since $\mathrm{Var}[X] = 0.16$, we must have

$$\frac{0.16}{100(0.05)^2} = \alpha \tag{4}$$

This reduces to $\alpha = 16/25 = 0.64$.

(d) Again we use Equation (3). To meet the specified requirement, we choose $c = 0.1$. Since $\text{Var}[X] = 0.16$, we must have

$$\frac{0.16}{n(0.1)^2} \leq 0.05 \tag{5}$$

The smallest value that meets the requirement is $n = 320$.

Quiz 10.4 Solution

Define the random variable $W = (X - \mu_X)^2$. Observe that $V_{100}(X) = M_{100}(W)$. By Theorem 10.10, the mean square error is

$$\text{E}\left[(M_{100}(W) - \mu_W)^2\right] = \frac{\text{Var}[W]}{100}. \tag{1}$$

Observe that $\mu_X = 0$ so that $W = X^2$. Thus,

$$\mu_W = \text{E}\left[X^2\right] = \int_{-1}^{1} x^2 f_X(x)\, dx = 1/3, \tag{2}$$

$$\text{E}\left[W^2\right] = \text{E}\left[X^4\right] = \int_{-1}^{1} x^4 f_X(x)\, dx = 1/5. \tag{3}$$

Therefore $\text{Var}[W] = \text{E}[W^2] - \mu_W^2 = 1/5 - (1/3)^2 = 4/45$ and the mean square error is $4/4500 = 0.0009$.

Quiz 10.5 Solution

Assuming the number n of samples is large, we can use a Gaussian approximation for $M_n(X)$. Since $\text{E}[X] = p$ and $\text{Var}[X] = p(1-p)$, we apply Theorem 10.14 which says that the interval estimate

$$M_n(X) - c \leq p \leq M_n(X) + c \tag{1}$$

has confidence coefficient $1 - \alpha$ where

$$\alpha = 2 - 2\Phi\left(\frac{c\sqrt{n}}{p(1-p)}\right). \tag{2}$$

We must ensure for every value of p that $1-\alpha \geq 0.9$ or $\alpha \leq 0.1$. Equivalently, we must have

$$\Phi\left(\frac{c\sqrt{n}}{p(1-p)}\right) \geq 0.95 \tag{3}$$

for every value of p. Since $\Phi(x)$ is an increasing function of x, we must satisfy $c\sqrt{n} \geq 1.65p(1-p)$. Since $p(1-p) \leq 1/4$ for all p, we require that

$$c \geq \frac{1.65}{4\sqrt{n}} = \frac{0.41}{\sqrt{n}}. \tag{4}$$

The 0.9 confidence interval estimate of p is

$$M_n(X) - \frac{0.41}{\sqrt{n}} \leq p \leq M_n(X) + \frac{0.41}{\sqrt{n}}.$$

For the 0.99 confidence interval, we have $\alpha \leq 0.01$, implying

$$\Phi(c\sqrt{n}/(p(1-p))) \geq 0.995. \tag{5}$$

This implies $c\sqrt{n} \geq 2.58p(1-p)$. Since $p(1-p) \leq 1/4$ for all p, we require that $c \geq (0.25)(2.58)/\sqrt{n}$. In this case, the 0.99 confidence interval estimate is

$$M_n(X) - \frac{0.645}{\sqrt{n}} \leq p \leq M_n(X) + \frac{0.645}{\sqrt{n}}. \tag{6}$$

Note that if $M_{100}(X) = 0.4$, then the 0.99 confidence interval estimate is

$$0.3355 \leq p \leq 0.4645. \tag{7}$$

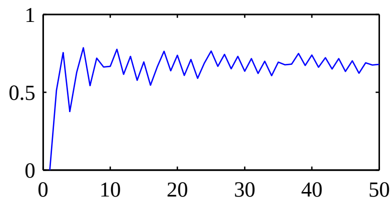
The interval is wide because the 0.99 confidence is high.

Quiz 10.6 Solution

Following the `bernoullitraces.m` approach, we generate $m = 1000$ sample paths, each sample path having $n = 100$ Bernoulli traces. at time k, `OK(k)` counts the fraction of sample paths that have sample mean within one standard error of p. The program `bernoullisample.m` generates graphs the number of traces within one standard error as a function of the time, i.e. the number of trials in each trace.

```
function OK=bernoullisample(n,m,p);
x=reshape(bernoullirv(p,m*n),n,m);
nn=(1:n)'*ones(1,m);
MN=cumsum(x)./nn;
stderr=sqrt(p*(1-p))./sqrt((1:n)');
stderrmat=stderr*ones(1,m);
OK=sum(abs(MN-p)<stderrmat,2)/m;
plot(1:n,OK);
```

The following graph was generated by `bernoullisample(50,5000,0.5)`:



As we would expect, as m gets large, the fraction of traces within one standard error approaches $2\Phi(1) - 1 \approx 0.68$. The unusual sawtooth pattern, though perhaps unexpected, is examined in Problem 10.6.2.

Quiz 11.1 Solution

From the problem statement, each X_i has CDF

$$F_{X_i}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases} \tag{1}$$

Hence, the CDF of the maximum of $X_1, \ldots, X_{15}$ obeys

$$\begin{aligned} F_X(x) &= \mathrm{P}[X \leq x] \\ &= \mathrm{P}[X_1 \leq x, \cdots, X_{15} \leq x] \\ &= [\mathrm{P}[X_i \leq x]]^{15}. \end{aligned} \tag{2}$$

This implies that for $x \geq 0$,

$$F_X(x) = [F_{X_i}(x)]^{15} = \left[1 - e^{-x}\right]^{15} \tag{3}$$

To design a significance test, we must choose a rejection region for X. A reasonable choice is to reject the hypothesis if X is too small. That is, let $R = \{X \leq r\}$. For a significance level of $\alpha = 0.01$, we obtain

$$\begin{aligned} \alpha = \mathrm{P}[X \leq r] &= (1 - e^{-r})^{15} \\ &= 0.01. \end{aligned} \tag{4}$$

It is straightforward to show that

$$r = -\ln\left[1 - (0.01)^{1/15}\right] = 1.33 \tag{5}$$

Hence, if we observe $X < 1.33$, then we reject the hypothesis.

Quiz 11.2 Solution

From the problem statement, the conditional PMFs of K are

$$P_{K|H_0}(k) = \begin{cases} \frac{10^{4k} e^{-10^4}}{k!} & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P_{K|H_1}(k) = \begin{cases} \frac{10^{6k} e^{-10^6}}{k!} & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Since the two hypotheses are equally likely, the MAP and ML tests are the same. From Theorem 11.6, the ML hypothesis rule is

$$k \in A_0 \text{ if } P_{K|H_0}(k) \geq P_{K|H_1}(k); \qquad k \in A_1 \text{ otherwise.} \tag{3}$$

This rule simplifies to

$$k \in A_0 \text{ if } k \leq k^* = \frac{10^6 - 10^4}{\ln 100} = 214,975.7; \qquad k \in A_1 \text{ otherwise.} \tag{4}$$

Thus if we observe at least 214,976 photons, then we accept hypothesis H_1.

Quiz 11.3 Solution

For the QPSK system, a symbol error occurs when s_i is transmitted but $(X_1, X_2) \in A_j$ for some $j \neq i$. For a QPSK system, it is easier to calculate the probability of a correct decision. Given H_0, the conditional probability of a correct decision is

$$\begin{aligned} \mathrm{P}[C|H_0] &= \mathrm{P}[X_1 > 0, X_2 > 0|H_0] \\ &= \mathrm{P}\left[\sqrt{E/2} + N_1 > 0, \sqrt{E/2} + N_2 > 0\right]. \end{aligned} \tag{1}$$

Because of the symmetry of the signals, $\mathrm{P}[C|H_0] = \mathrm{P}[C|H_i]$ for all i. This implies the probability of a correct decision is $\mathrm{P}[C] = \mathrm{P}[C|H_0]$. Since N_1 and N_2 are iid Gaussian $(0, \sigma)$ random variables, we have

$$\begin{aligned} \mathrm{P}[C] = \mathrm{P}[C|H_0] &= \mathrm{P}\left[\sqrt{E/2} + N_1 > 0\right] \mathrm{P}\left[\sqrt{E/2} + N_2 > 0\right] \\ &= \left(\mathrm{P}\left[N_1 > -\sqrt{E/2}\right]\right)^2 \\ &= \left[1 - \Phi\left(\frac{-\sqrt{E/2}}{\sigma}\right)\right]^2. \end{aligned} \tag{2}$$

Since $\Phi(-x) = 1 - \Phi(x)$, we have $\mathrm{P}[C] = \Phi^2(\sqrt{E/2\sigma^2})$. Equivalently, the probability of error is

$$P_{\mathrm{ERR}} = 1 - \mathrm{P}[C] = 1 - \Phi^2\left(\sqrt{\frac{E}{2\sigma^2}}\right). \tag{3}$$

Quiz 11.4 Solution

To generate the ROC, the existing program `sqdistor` already calculates this miss probability $P_{\text{MISS}} = P_{01}$ and the false alarm probability $P_{\text{FA}} = P_{10}$. The modified program, `sqdistroc.m` is essentially the same as `sqdistor` except the output is a matrix `FM` whose columns are the false alarm and miss probabilities. Here is the modified code:

```
function FM=sqdistroc(v,d,m,T)
%square law distortion recvr
%P(error) for m bits tested
%transmit v volts or -v volts,
%add N volts, N is Gauss(0,1)
%add d(v+N)^2 distortion
%receive 1 if x>T, otherwise 0
%FM = [P(FA) P(MISS)]
x=(v+randn(m,1));
[XX,TT]=ndgrid(x,T(:));
P01=sum((XX+d*(XX.^2)< TT),1)/m;
x= -v+randn(m,1);
[XX,TT]=ndgrid(x,T(:));
P10=sum((XX+d*(XX.^2)>TT),1)/m;
FM=[P10(:) P01(:)];
```

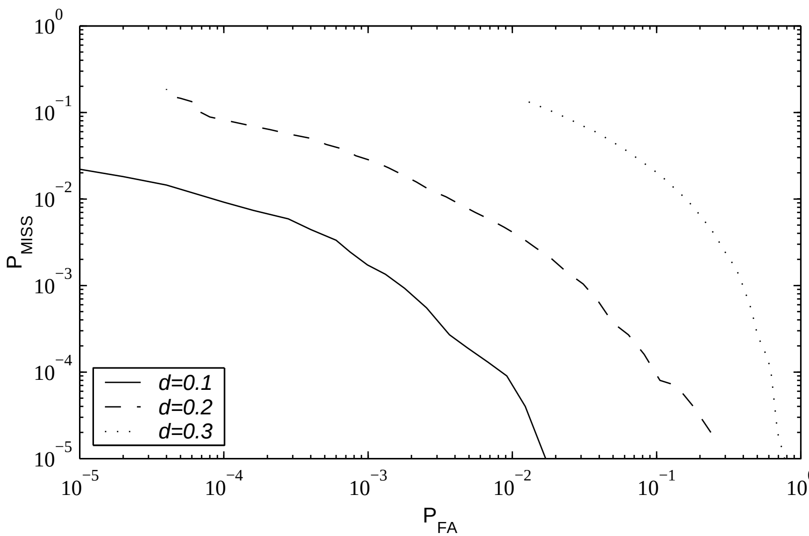
Next, the program `sqdistrocplot.m` calls `sqdistroc` three times to generate a plot that compares the receiver performance for the three requested values of d.

```
function FM=sqdistrocplot(v,m,T);
FM1=sqdistroc(v,0.1,m,T);
FM2=sqdistroc(v,0.2,m,T);
FM5=sqdistroc(v,0.3,m,T);
FM=[FM1 FM2 FM5];
loglog(FM1(:,1),FM1(:,2),'-k',FM2(:,1),FM2(:,2),'--k',...
   FM5(:,1),FM5(:,2),':k');
legend('\it d=0.1','\it d=0.2','\it d=0.3',3)
ylabel('P_{MISS}');
xlabel('P_{FA}');
```

To see the effect of d, the commands

```
T=-3:0.1:3; sqdistrocplot(3,100000,T);
```

generated this receiver operating curve:



`T=-3:0.1:3; sqdistrocplot(3,100000,T);`

Quiz 12.1 Solution

(a) First, we calculate the marginal PDF for $0 \leq y \leq 1$:

$$f_Y(y) = \int_0^y 2(y+x)\,dx = 2xy + x^2\Big|_{x=0}^{x=y} = 3y^2. \tag{1}$$

This implies the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{3y} + \frac{2x}{3y^2} & 0 \leq x \leq y, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The minimum mean square error estimate of X given $Y = y$ is

$$\hat{x}_M(y) = \mathrm{E}\left[X|Y=y\right] = \int_0^y \left(\frac{2x}{3y} + \frac{2x^2}{3y^2}\right) dx = 5y/9. \tag{3}$$

Thus the MMSE estimator of X given Y is $\hat{X}_M(Y) = 5Y/9$.

(c) To obtain the conditional PDF $f_{Y|X}(y|x)$, we need the marginal PDF $f_X(x)$. For $0 \leq x \leq 1$,

$$f_X(x) = \int_x^1 2(y+x)\,dy = y^2 + 2xy\Big|_{y=x}^{y=1} = 1 + 2x - 3x^2. \tag{4}$$

For $0 \leq x \leq 1$, the conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2(y+x)}{1+2x-3x^2} & x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

(d) The MMSE estimate of Y given $X = x$ is

$$\hat{y}_M(x) = \mathrm{E}\left[Y|X=x\right] = \int_x^1 \frac{2y^2 + 2xy}{1+2x-3x^2}\,dy \tag{6}$$

$$= \frac{2y^3/3 + xy^2}{1+2x-3x^2}\Bigg|_{y=x}^{y=1} \tag{7}$$

$$= \frac{2+3x-5x^3}{3+6x-9x^2}. \tag{8}$$

Quiz 12.2 Solution

(a) Since the expectation of the sum equals the sum of the expectations,

$$\mathrm{E}\left[R\right] = \mathrm{E}\left[T\right] + \mathrm{E}\left[X\right] = 0. \tag{1}$$

(b) Since T and X are independent, the variance of the sum $R = T + X$ is

$$\mathrm{Var}[R] = \mathrm{Var}[T] + \mathrm{Var}[X] = 9 + 3 = 12. \tag{2}$$

(c) Since T and R have expected values $\mathrm{E}[R] = \mathrm{E}[T] = 0$,

$$\mathrm{Cov}\left[T, R\right] = \mathrm{E}\left[TR\right] = \mathrm{E}\left[T(T + X)\right] = \mathrm{E}\left[T^2\right] + \mathrm{E}\left[TX\right]. \tag{3}$$

Since T and X are independent and have zero expected value, $\mathrm{E}[TX] = \mathrm{E}[T]\,\mathrm{E}[X] = 0$ and $\mathrm{E}[T^2] = \mathrm{Var}[T]$. Thus $\mathrm{Cov}[T, R] = \mathrm{Var}[T] = 9$.

(d) From Definition 5.6, the correlation coefficient of T and R is

$$\rho_{T,R} = \frac{\mathrm{Cov}\left[T, R\right]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[T]}} = \frac{\sigma_T}{\sigma_R} = \sqrt{3}/2. \tag{4}$$

(e) From Theorem 12.3, the optimum linear estimate of T given R is

$$\hat{T}_L(R) = \rho_{T,R}\frac{\sigma_T}{\sigma_R}\left(R - \mathrm{E}\left[R\right]\right) + \mathrm{E}\left[T\right]. \tag{5}$$

Since $\mathrm{E}[R] = \mathrm{E}[T] = 0$ and $\rho_{T,R} = \sigma_T/\sigma_R$,

$$\hat{T}_L(R) = \frac{\sigma_T^2}{\sigma_R^2}R = \frac{\sigma_T^2}{\sigma_T^2 + \sigma_X^2}R = \frac{3}{4}R. \tag{6}$$

Hence $a^* = 3/4$ and $b^* = 0$.

(f) By Theorem 12.3, the mean square error of the linear estimate is

$$e_L^* = \mathrm{Var}[T](1 - \rho_{T,R}^2) = 9(1 - 3/4) = 9/4. \tag{7}$$

Quiz 12.3 Solution

When $R = r$, the conditional PDF of $X = Y - 40 - 40\log_{10} r$ is Gaussian with expected value $-40 - 40\log_{10} r$ and variance 64. The conditional PDF of X given R is

$$f_{X|R}(x|r) = \frac{1}{\sqrt{128\pi}} e^{-(x+40+40\log_{10} r)^2/128}. \tag{1}$$

From the conditional PDF $f_{X|R}(x|r)$, we can use Definition 12.2 to write the ML estimate of R given $X = x$ as

$$\hat{r}_{\text{ML}}(x) = \arg\max_{r\geq 0} f_{X|R}(x|r)\,. \tag{2}$$

We observe that $f_{X|R}(x|r)$ is maximized when the exponent $(x+40+40\log_{10} r)^2$ is minimized. This minimum occurs when the exponent is zero, yielding

$$\log_{10} r = -1 - x/40 \tag{3}$$

or

$$\hat{r}_{\text{ML}}(x) = (0.1)10^{-x/40} \text{ m}. \tag{4}$$

If the result doesn't look correct, note that a typical figure for the signal strength might be $x = -120$ dB. This corresponds to a distance estimate of $\hat{r}_{\text{ML}}(-120) = 100$ m.

For the MAP estimate, we observe that the joint PDF of X and R is

$$f_{X,R}(x,r) = f_{X|R}(x|r)\, f_R(r) = \frac{1}{10^6\sqrt{32\pi}} r e^{-(x+40+40\log_{10} r)^2/128}. \tag{5}$$

From Theorem 12.5, the MAP estimate of R given $X = x$ is the value of r that maximizes $f_{X,R}(x,r)$. That is,

$$\hat{r}_{\text{MAP}}(x) = \arg\max_{0\leq r\leq 1000} f_{X,R}(x,r)\,. \tag{6}$$

Note that we have included the constraint $r \leq 1000$ in the maximization to highlight the fact that under our probability model, $R \leq 1000$ m. Setting the derivative of $f_{X,R}(x,r)$ with respect to r to zero yields

$$e^{-(x+40+40\log_{10} r)^2/128}\left[1 - \frac{80\log_{10} e}{128}(x + 40 + 40\log_{10} r)\right] = 0. \tag{7}$$

Solving for r yields

$$r = 10^{\left(\frac{1}{25\log_{10} e} - 1\right)} 10^{-x/40} = (0.1236)10^{-x/40}. \tag{8}$$

This is the MAP estimate of R given $X = x$ as long as $r \leq 1000$ m. When $x \leq -156.3$ dB, the above estimate will exceed 1000 m, which is not possible in our probability model. Hence, the complete description of the MAP estimate is

$$\hat{r}_{\text{MAP}}(x) = \begin{cases} 1000 & x < -156.3, \\ (0.1236)10^{-x/40} & x \geq -156.3. \end{cases} \tag{9}$$

For example, if $x = -120$dB, then $\hat{r}_{\text{MAP}}(-120) = 123.6$ m. When the measured signal strength is not too low, the MAP estimate is 23.6% larger than the ML estimate. This reflects the fact that large values of R are a priori more probable than small values. However, for very low signal strengths, the MAP estimate takes into account that the distance can never exceed 1000 m.

Quiz 12.4 Solution

(a) From Theorem 12.3, the LMSE estimate of X_2 given Y_2 is

$$\begin{aligned} \hat{X}_2(Y_2) &= \rho_{X_2,Y_2}\frac{\sigma_{X_2}}{\sigma_{Y_2}}(Y_2 - \mu_{Y_2}) + \mu_{X_2} \\ &= a^*Y_2 + b^*, \end{aligned} \tag{1}$$

where

$$a^* = \rho_{X,Y}\frac{\sigma_X}{\sigma_Y} = \frac{\mathrm{Cov}\left[X_2, Y_2\right]}{\mathrm{Var}[Y_2]}, \qquad b^* = \mu_{X_2} - a^*\mu_{Y_2}. \tag{2}$$

Because $\mathrm{E}[\mathbf{X}] = \mathrm{E}[\mathbf{Y}] = \mathbf{0}$,

$$\mathrm{Cov}\left[X_2, Y_2\right] = \mathrm{E}\left[X_2 Y_2\right] = \mathrm{E}\left[X_2(X_2 + W_2)\right] = \mathrm{E}\left[X_2^2\right] = 1, \tag{3}$$
$$\mathrm{Var}[Y_2] = \mathrm{Var}[X_2] + \mathrm{Var}[W_2] = \mathrm{E}\left[X_2^2\right] + \mathrm{E}\left[W_2^2\right] = 1.1. \tag{4}$$

It follows that $a^* = 1/1.1$. Because $\mu_{X_2} = \mu_{Y_2} = 0$, it follows that $b^* = 0$. Finally, to compute the expected square error, we calculate the correlation coefficient

$$\rho_{X_2,Y_2} = \frac{\mathrm{Cov}\left[X_2, Y_2\right]}{\sigma_{X_2}\sigma_{Y_2}} = \frac{1}{\sqrt{1.1}}. \tag{5}$$

The expected square error is

$$e_L^* = \mathrm{Var}[X_2](1 - \rho_{X_2,Y_2}^2) = 1 - \frac{1}{1.1} = \frac{1}{11} = 0.0909. \tag{6}$$

(b) Here we wish to estimate X_2 given the observation vector $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$. Since $\mathbf{Y} = \mathbf{X} + \mathbf{W}$ and $\mathrm{E}[\mathbf{X}] = \mathrm{E}[\mathbf{W}] = \mathbf{0}$, it follows that $\mathrm{E}[\mathbf{Y}] = \mathbf{0}$. Thus we can apply Theorem 12.6 and write the minimum mean square error linear estimator as

$$\hat{X}_2(\mathbf{Y}) = \mathbf{R}_{X_2\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y}. \tag{7}$$

We need to find $\mathbf{R}_{\mathbf{Y}}$ and $\mathbf{R}_{X_2\mathbf{Y}}$. Note that $\mathbf{X}$ and $\mathbf{W}$ have correlation matrices

$$\mathbf{R}_{\mathbf{X}} = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 1 \end{bmatrix}, \qquad \mathbf{R}_{\mathbf{W}} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{8}$$

This implies

$$\begin{aligned}\mathbf{R_Y} = \mathrm{E}\left[\mathbf{YY}'\right] &= \mathrm{E}\left[(\mathbf{X}+\mathbf{W})(\mathbf{X}'+\mathbf{W}')\right] \\ &= \mathrm{E}\left[\mathbf{XX}'+\mathbf{XW}'+\mathbf{WX}'+\mathbf{WW}'\right]. \end{aligned} \tag{9}$$

Because $\mathbf{X}$ and $\mathbf{W}$ are independent, $\mathrm{E}[\mathbf{XW}'] = \mathrm{E}[\mathbf{X}]\,\mathrm{E}[\mathbf{W}'] = \mathbf{0}$. Similarly, $\mathrm{E}[\mathbf{WX}'] = \mathbf{0}$. This implies

$$\mathbf{R_Y} = \mathrm{E}\left[\mathbf{XX}'\right] + \mathrm{E}\left[\mathbf{WW}'\right] = \mathbf{R_X} + \mathbf{R_W} = \begin{bmatrix} 1.1 & -0.9 \\ -0.9 & 1.1 \end{bmatrix}. \tag{10}$$

In addition, we need to find

$$\begin{aligned}\mathbf{R}_{X_2\mathbf{Y}} = \mathrm{E}\left[X_2\mathbf{Y}'\right] &= \begin{bmatrix}\mathrm{E}\left[X_2Y_1\right] & \mathrm{E}\left[X_2Y_2\right]\end{bmatrix} \\ &= \begin{bmatrix}\mathrm{E}\left[X_2(X_1+W_1)\right] & \mathrm{E}\left[X_2(X_2+W_2)\right]\end{bmatrix}. \end{aligned} \tag{11}$$

Since $\mathbf{X}$ and $\mathbf{W}$ are independent vectors, $\mathrm{E}[X_2W_1] = \mathrm{E}[X_2]\,\mathrm{E}[W_1] = 0$ and $\mathrm{E}[X_2W_2] = 0$. Thus

$$\mathbf{R}_{X_2\mathbf{Y}} = \begin{bmatrix}\mathrm{E}[X_1X_2] & \mathrm{E}\left[X_2^2\right]\end{bmatrix} = \begin{bmatrix}-0.9 & 1\end{bmatrix}. \tag{12}$$

It follows that

$$\begin{aligned}\mathbf{R}_{X_2\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1} &= \begin{bmatrix}-0.9 & 1\end{bmatrix}\begin{bmatrix} 1.1 & -0.9 \\ -0.9 & 1.1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix}-0.9 & 1\end{bmatrix}\begin{bmatrix} 2.75 & 2.25 \\ 2.25 & 2.75 \end{bmatrix} = \begin{bmatrix}-0.225 & 0.725\end{bmatrix}. \end{aligned} \tag{13}$$

Therefore, the optimum linear estimator of X_2 given Y_1 and Y_2 is

$$\hat{X}_2(\mathbf{Y}) = \mathbf{R}_{X_2\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y} = -0.225Y_1 + 0.725Y_2. \tag{14}$$

From Theorem 12.6(b), the mean square error is

$$\begin{aligned} e_2^* &= \mathrm{Var}\left[X_2\right] - \mathbf{R}_{X_2\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{R}'_{X_2\mathbf{Y}} \\ &= 1 - \begin{bmatrix}-0.225 & 0.725\end{bmatrix}\begin{bmatrix}-0.9 \\ 1\end{bmatrix} = 0.0725. \end{aligned} \tag{15}$$

In part (a), we used only Y_2 to estimate X_2 and the resulting mean square error was 0.0909. Here we showed that by using both Y_1 and Y_2 to estimate X_2, we reduced the mean square error to 0.0725, about a 20% reduction.

Quiz 12.5 Solution

Since X and $\mathbf{W}$ have zero expected value, $\mathbf{Y}$ also has zero expected value. Thus, by Theorem 12.6,

$$\hat{X}_L(\mathbf{Y}) = \mathbf{R}_{X\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y} \tag{1}$$

Since X and $\mathbf{W}$ are independent, $\mathrm{E}[\mathbf{W}X] = \mathbf{0}$ and $\mathrm{E}[X\mathbf{W}'] = \mathbf{0}'$. This implies

$$\mathbf{R}_{X\mathbf{Y}} = \mathrm{E}\left[X\mathbf{Y}'\right] = \mathrm{E}\left[X(\mathbf{1}'X + \mathbf{W}')\right] = \mathbf{1}'\,\mathrm{E}\left[X^2\right] = \mathbf{1}'. \tag{2}$$

Note that $\mathbf{1}'$ is the row vector $\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}$ of twenty ones. By the same reasoning, the correlation matrix of $\mathbf{Y}$ is

$$\begin{aligned}
\mathbf{R}_{\mathbf{Y}} = \mathrm{E}\left[\mathbf{Y}\mathbf{Y}'\right] &= \mathrm{E}\left[(\mathbf{1}X + \mathbf{W})(\mathbf{1}'X + \mathbf{W}')\right] && (3)\\
&= \mathbf{1}\mathbf{1}'\,\mathrm{E}\left[X^2\right] + \mathbf{1}\,\mathrm{E}\left[X\mathbf{W}'\right] + \mathrm{E}\left[\mathbf{W}X\right]\mathbf{1}' + \mathrm{E}\left[\mathbf{W}\mathbf{W}'\right] && (4)\\
&= \mathbf{1}\mathbf{1}' + \mathbf{R}_{\mathbf{W}} && (5)
\end{aligned}$$

Note that $\mathbf{1}\mathbf{1}'$ is a 20×20 matrix with every entry equal to 1. Thus,

$$\mathbf{R}_{X\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1} = \mathbf{1}'\left(\mathbf{1}\mathbf{1}' + \mathbf{R}_W\right)^{-1} \tag{6}$$

and the optimal linear estimator is

$$\hat{X}_L(\mathbf{Y}) = \mathbf{1}'\left(\mathbf{1}\mathbf{1}' + \mathbf{R}_{\mathbf{W}}\right)^{-1}\mathbf{Y}. \tag{7}$$

By Theorem 12.6(b), the mean square error is

$$e_L^* = \mathrm{Var}[X] - \mathbf{R}_{X\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{R}_{\mathbf{Y}X} = 1 - \mathbf{1}'\left(\mathbf{1}\mathbf{1}' + \mathbf{R}_{\mathbf{W}}\right)^{-1}\mathbf{1}. \tag{8}$$

Now we note that $\mathbf{R}_W$ has i,jth entry $R_{\mathbf{W}}(i,j) = c^{|i-j|-1}$. The question we must address is what value c minimizes e_L^*. This problem is atypical in that we do not usually get to choose the correlation structure of the noise. However, we will see that the answer is somewhat instructive.

We note that the optimal c is not obviously apparent from Equation (8). In particular, we observe that $\mathrm{Var}[W_i] = R_{\mathbf{W}}(i,i) = 1/c$. Thus, when c is small, the noises W_i have high variance and we would expect our estimator to be poor. On the other hand, if c is large then W_i and W_j are highly correlated and the separate measurements of X are very dependent. This would suggest that large values of c will also result in poor MSE. If this argument is not clear, consider the extreme case in which every W_i and W_j have correlation coefficient $\rho_{ij} = 1$. In this case, our 20 measurements will be all the same and one measurement is as good as 20 measurements.

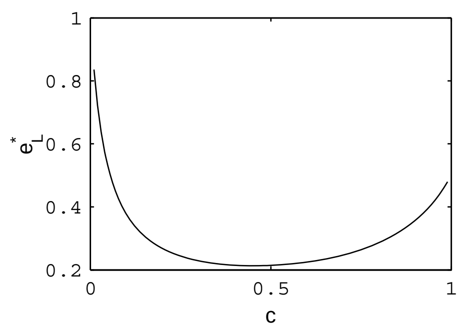
To find the optimal value of c, we write a MATLAB function `mquiz9(c)` to calculate the MSE for a given c and second function that finds the MSE for a range of values of c.

```
function [mse,af]=mquiz9(c);
v1=ones(20,1);
RW=toeplitz(c.^((0:19)-1));
RY=(v1*(v1')) +RW;
af=(inv(RY))*v1;
mse=1-((v1')*af);
```

```
function cmin=mquiz9minc(c);
msec=zeros(size(c));
for k=1:length(c),
    [msec(k),af]=mquiz9(c(k));
end
plot(c,msec);
xlabel('c');ylabel('e_L^*');
[msemin,optk]=min(msec);
cmin=c(optk);
```

Note in `mquiz9` that `v1` corresponds to the vector $\mathbf{1}$ of all ones. The following commands finds the minimum c and also produces the following graph:

```
>> c=0.01:0.01:0.99;
>> mquiz9minc(c)
ans =
    0.4500
```



As we see in the graph, both small values and large values of c result in large MSE.

Quiz 13.1 Solution

(a) We obtain a continuous-time, continuous-value process when we record the temperature as a continuous waveform over time.

(b) If at every moment in time, we round the temperature to the nearest degree, then we obtain a continuous-time, discrete-value process.

(c) If we sample the process in part (a) every T seconds, then we obtain a discrete-time, continuous-value process.

(d) Rounding the samples in part (c) to the nearest integer degree yields a discrete-time, discrete-value process.

Quiz 13.2 Solution

(a) Each resistor has resistance R in ohms with uniform PDF

$$f_R(r) = \begin{cases} 0.01 & 950 \leq r \leq 1050 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The probability that a test produces a 1% resistor is

$$p = \mathrm{P}\left[990 \leq R \leq 1010\right] = \int_{990}^{1010} (0.01)\, dr = 0.2. \tag{2}$$

(b) In t seconds, exactly t resistors are tested. Each resistor is a 1% resistor with probability $p = 0.2$, independent of any other resistor. Consequently, the number of 1% resistors found has the binomial $(t, 0.2)$ PMF

$$P_{N(t)}(n) = \binom{t}{n} (0.2)^n (0.8)^{t-n}. \tag{3}$$

(c) First we will find the PMF of T_1. This problem is easy if we view each resistor test as an independent trial. A success occurs on a trial with probability $p = 0.2$ if we find a 1% resistor. The first 1% resistor is found at time $T_1 = t$ if we observe failures on trials $1, \ldots, t-1$ followed by a success on trial t. Hence, just as in Example 3.9, T_1 has the geometric (0.2) PMF

$$P_{T_1}(t) = \begin{cases} (0.8)^{t-1}(0.2) & t = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

From Theorem 3.5, a geometric random variable with success probability p has expected value $1/p$. In this problem, $\mathrm{E}[T_1] = 1/p = 5$.

(d) Since $p = 0.2$, the probability the first 1% resistor is found in exactly five seconds is $P_{T_1}(5) = (0.8)^4(0.2) = 0.08192$.

(e) Note that once we find the first 1% resistor, the number of additional trials needed to find the second 1% resistor once again has a geometric PMF with expected value $1/p$ since each independent trial is a success with probability p. That is, $T_2 = T_1 + T'$ where T' is independent and identically distributed to T_1. Thus

$$\begin{aligned} \mathrm{E}\left[T_2 | T_1 = 10\right] &= \mathrm{E}\left[T_1 | T_1 = 10\right] + \mathrm{E}\left[T' | T_1 = 10\right] \\ &= 10 + \mathrm{E}\left[T'\right] = 10 + 5 = 15. \end{aligned} \tag{5}$$

Quiz 13.3 Solution

Since each X_i is a $N(0,1)$ random variable, each X_i has PDF

$$f_{X_i)}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{1}$$

By Theorem 13.1, the joint PDF of $\mathbf{X} = \begin{bmatrix} X_1 & \cdots X_n \end{bmatrix}'$ is

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X(1),\ldots,X(n)}(x_1, \ldots, x_n) = \prod_{i=1}^{k} f_X(x_i) = \frac{1}{(2\pi)^{n/2}} e^{-(x_1^2 + \cdots + x_n^2)/2}. \tag{2}$$

Quiz 13.4 Solution

The first and second hours are nonoverlapping intervals. Since one hour equals 3600 sec and the Poisson process has a rate of 10 packets/sec, the expected number of packets in each hour is $E[M_i] = \alpha = 36,000$. This implies M_1 and M_2 are independent Poisson random variables each with PMF

$$P_{M_i}(m) = \begin{cases} \frac{\alpha^m e^{-\alpha}}{m!} & m = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since M_1 and M_2 are independent, the joint PMF of M_1 and M_2 is

$$P_{M_1,M_2}(m_1, m_2) = P_{M_1}(m_1)\, P_{M_2}(m_2) = \begin{cases} \frac{\alpha^{m_1+m_2} e^{-2\alpha}}{m_1! m_2!} & \begin{array}{l} m_1 = 0, 1, \ldots; \\ m_2 = 0, 1, \ldots, \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Quiz 13.5 Solution

To answer whether $N'(t)$ is a Poisson process, we look at the interarrival times. Let $X_1, X_2, \ldots$ denote the interarrival times of the $N(t)$ process. Since we count only even-numbered arrival for $N'(t)$, the time until the first arrival of the $N'(t)$ is $Y_1 = X_1 + X_2$. Since X_1 and X_2 are independent exponential (λ) random variables, Y_1 is an Erlang $(n = 2, \lambda)$ random variable; see Theorem 9.9. Since $Y_i(t)$, the ith interarrival time of the $N'(t)$ process, has the same PDF as $Y_1(t)$, we can conclude that the interarrival times of $N'(t)$ are not exponential random variables. Thus $N'(t)$ is *not* a Poisson process.

Quiz 13.6 Solution

First, we note that for $t > s$,

$$X(t) - X(s) = \frac{W(t) - W(s)}{\sqrt{\alpha}}. \tag{1}$$

Since $W(t) - W(s)$ is a Gaussian random variable, Theorem 4.13 states that $W(t) - W(s)$ is Gaussian with expected value

$$\mathrm{E}\left[X(t) - X(s)\right] = \frac{\mathrm{E}\left[W(t) - W(s)\right]}{\sqrt{\alpha}} = 0 \tag{2}$$

and variance

$$\mathrm{E}\left[(W(t) - W(s))^2\right] = \frac{\mathrm{E}\left[(W(t) - W(s))^2\right]}{\alpha} = \frac{\alpha(t-s)}{\alpha}. \tag{3}$$

Consider $s' \leq s < t$. Since $s \geq s'$, $W(t) - W(s)$ is independent of $W(s')$. This implies $[W(t) - W(s)]/\sqrt{\alpha}$ is independent of $W(s')/\sqrt{\alpha}$ for all $s \geq s'$. That is, $X(t) - X(s)$ is independent of $X(s')$ for all $s \geq s'$. Thus $X(t)$ is a Brownian motion process with variance $\mathrm{Var}[X(t)] = t$.

Quiz 13.7 Solution

First we find the expected value

$$\mu_Y(t) = \mu_X(t) + \mu_N(t) = \mu_X(t). \tag{1}$$

To find the autocorrelation, we observe that since $X(t)$ and $N(t)$ are independent and since $N(t)$ has zero expected value, $\mathrm{E}[X(t)N(t')] = \mathrm{E}[X(t)]\,\mathrm{E}[N(t')] = 0$. Since $R_Y(t,\tau) = \mathrm{E}[Y(t)Y(t+\tau)]$, we have

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}\left[(X(t) + N(t))\left(X(t+\tau) + N(t+\tau)\right)\right] \\ &= \mathrm{E}\left[X(t)X(t+\tau)\right] + \mathrm{E}\left[X(t)N(t+\tau)\right] \\ &\qquad + \mathrm{E}\left[X(t+\tau)N(t)\right] + \mathrm{E}\left[N(t)N(t+\tau)\right] \\ &= R_X(t,\tau) + R_N(t,\tau). \end{aligned} \tag{2}$$

Quiz 13.8 Solution

From Definition 13.14, $X_1, X_2, \ldots$ is a stationary random sequence if for all sets of time instants $n_1, \ldots, n_m$ and time offset k,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1,\ldots,x_m) = f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1,\ldots,x_m). \tag{1}$$

Since the random sequence is iid,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1,\ldots,x_m) = f_X(x_1)\, f_X(x_2) \cdots f_X(x_m)\,. \tag{2}$$

Similarly, for time instants $n_1 + k, \ldots, n_m + k$,

$$f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1,\ldots,x_m) = f_X(x_1)\, f_X(x_2) \cdots f_X(x_m)\,. \tag{3}$$

We can conclude that the iid random sequence is stationary.

Quiz 13.9 Solution

We must check whether each function $R(\tau)$ meets the conditions of Theorem 13.12:

$$R(\tau) \geq 0, \qquad R(\tau) = R(-\tau), \qquad |R(\tau)| \leq R(0). \tag{1}$$

(a) $R_1(\tau) = e^{-|\tau|}$ meets all three conditions and thus is valid.

(b) $R_2(\tau) = e^{-\tau^2}$ also is valid.

(c) $R_3(\tau) = e^{-\tau} \cos \tau$ is not valid because

$$R_3(-2\pi) = e^{2\pi} \cos 2\pi = e^{2\pi} > 1 = R_3(0) \tag{2}$$

(d) $R_4(\tau) = e^{-\tau^2} \sin \tau$ also cannot be an autocorrelation function because

$$R_4(\pi/2) = e^{-\pi/2} \sin \pi/2 = e^{-\pi/2} > 0 = R_4(0) \tag{3}$$

Quiz 13.10 Solution

(a) The autocorrelation of $Y(t)$ is

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}\left[Y(t)Y(t+\tau)\right] \\ &= \mathrm{E}\left[X(-t)X(-t-\tau)\right] \\ &= R_X(-t-(-t-\tau)) = R_X(\tau). \end{aligned} \tag{1}$$

Since $\mathrm{E}[Y(t)] = \mathrm{E}[X(-t)] = \mu_X$, we can conclude that $Y(t)$ is a wide sense stationary process. In fact, we see that by viewing a process backwards in time, we see the same second order statistics.

(b) Since $X(t)$ and $Y(t)$ are both wide sense stationary processes, we can check whether they are jointly wide sense stationary by seeing if $R_{XY}(t,\tau)$ is just a function of τ. In this case,

$$\begin{aligned} R_{XY}(t,\tau) &= \mathrm{E}\left[X(t)Y(t+\tau)\right] \\ &= \mathrm{E}\left[X(t)X(-t-\tau)\right] \\ &= R_X(t-(-t-\tau)) = R_X(2t+\tau). \end{aligned} \tag{2}$$

Since $R_{XY}(t,\tau)$ depends on both t and τ, we conclude that $X(t)$ and $Y(t)$ are not jointly wide sense stationary. To see why this is, suppose $R_X(\tau) = e^{-|\tau|}$ so that samples of $X(t)$ far apart in time have almost no correlation. In this case, as t gets larger, $Y(t) = X(-t)$ and $X(t)$ become less correlated.

Quiz 13.11 Solution

From the problem statement,

$$\mathrm{E}\left[X(t)\right] = \mathrm{E}\left[X(t+1)\right] = 0, \tag{1}$$

$$\mathrm{E}\left[X(t)X(t+1)\right] = 1/2, \tag{2}$$

$$\mathrm{Var}[X(t)] = \mathrm{Var}[X(t+1)] = 1. \tag{3}$$

The Gaussian random vector $\mathbf{X} = \begin{bmatrix} X(t) & X(t+1) \end{bmatrix}'$ has covariance matrix and corresponding inverse

$$\mathbf{C_X} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \qquad \mathbf{C_X}^{-1} = \frac{4}{3}\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}. \tag{4}$$

Since

$$\mathbf{x}'\mathbf{C_X}^{-1}\mathbf{x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}' \frac{4}{3}\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \frac{4}{3}\left(x_0^2 - x_0 x_+ x_1^2\right), \tag{5}$$

the joint PDF of $X(t)$ and $X(t+1)$ is the Gaussian vector PDF

$$\begin{aligned} f_{X(t),X(t+1)}(x_0,x_1) &= \frac{1}{(2\pi)^{n/2}[\det(\mathbf{C_X})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}'\mathbf{C_X}^{-1}\mathbf{x}\right) && (6) \\ &= \frac{1}{\sqrt{3\pi^2}} e^{-\frac{2}{3}\left(x_0^2 - x_0 x_1 + x_1^2\right)}. && (7) \end{aligned}$$

Quiz 13.12 Solution

The simple structure of the switch simulation of Example 13.28 admits a deceptively simple solution in terms of the vector of arrivals `A` and the vector of departures `D`. With the introduction of call blocking. we cannot generate these vectors all at once. In particular, when an arrival occurs at time t, we need to know that $M(t)$, the number of ongoing calls, satisfies $M(t) < c = 120$. Otherwise, when $M(t) = c$, we must block the call. Call blocking can be implemented by setting the service time of the call to zero so that the call departs as soon as it arrives.

The blocking switch is an example of a discrete event system. The system evolves via a sequence of discrete events, namely arrivals and departures, at discrete time instances. A simulation of the system moves from one time instant to the next by maintaining a chronological schedule of future events (arrivals and departures) to be executed. The program simply executes the event at the head of the schedule. The logic of such a simulation is

1. Start at time $t = 0$ with an empty system. Schedule the first arrival to occur at S_1, an exponential (λ) random variable.

2. Examine the head-of-schedule event.

   - When the head-of-schedule event is the kth arrival is at time t, check the state $M(t)$.
     - If $M(t) < c$, admit the arrival, increase the system state n by 1, and schedule a departure to occur at time $t + S_n$, where S_k is an exponential (λ) random variable.
     - If $M(t) = c$, block the arrival, do not schedule a departure event.
   - If the head of schedule event is a departure, reduce the system state n by 1.

3. Delete the head-of-schedule event and go to step 2.

After the head-of-schedule event is completed and any new events (departures in this system) are scheduled, we know the system state cannot change until the next scheduled event. Thus we know that $M(t)$ will stay the same until then. In our simulation, we use the vector `t` as the set of time instances at which we inspect the system state. Thus for all times `t(i)` between the current head-of-schedule event and the next, we set `m(i)` to the current switch state. Here is the complete program:

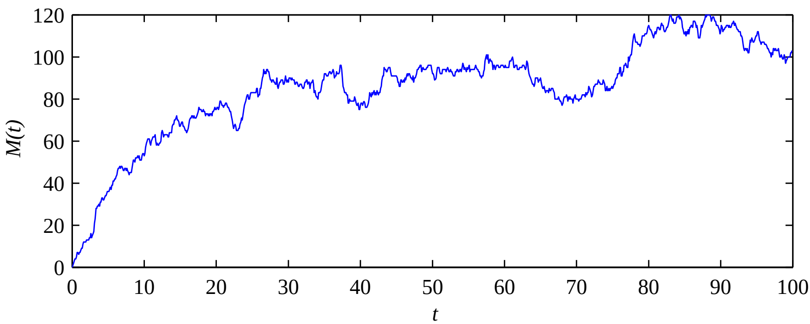
```
function [M,admits,blocks]=simblockswitch(lam,mu,c,t);
blocks=0; admits=0; %total no. blocks and admits
M=zeros(size(t)); n=0; % no. in system
time=[ exponentialrv(lam,1) ]; timenow=0; tmax=max(t);
event=[ 1 ]; %first event is an arrival
while (timenow<tmax)
   M((timenow<=t)&(t<time(1)))=n;
   timenow=time(1); eventnow=event(1);
   event(1)=[ ]; time(1)= [ ]; % clear current event
   if (eventnow==1) % arrival
       arrival=timenow+exponentialrv(lam,1); % next arrival
       b4arrival=time<arrival;
       event=[event(b4arrival) 1 event(~b4arrival)];
       time=[time(b4arrival) arrival time(~b4arrival)];
       if n<c %call admitted
         admits=admits+1; n=n+1;
         depart=timenow+exponentialrv(mu,1);
         b4depart=time<depart;
         event=[event(b4depart) -1 event(~b4depart)];
         time=[time(b4depart) depart time(~b4depart)];
       else
         blocks=blocks+1; % one more block, immed departure
         disp(sprintf('Time %10.3d Admits %10d Blocks %10d',...
             timenow,admits,blocks));
       end
   elseif (eventnow==-1) %departure
       n=n-1;
   end
end
```

In most programming languages, it is common to implement the event schedule as a linked list where each item in the list has a data structure indicating an event timestamp and the type of the event. In MATLAB, a simple (but not elegant) way to do this is to have maintain two vectors: `time` is a list of timestamps of scheduled events and `event` is a the list of event types. In this case, `event(i)=1` if the ith scheduled event is an arrival, or `event(i)=-1` if the ith scheduled event is a departure.

When the program is passed a vector `t`, the output `[m a b]` is such that `m(i)` is the number of ongoing calls at time `t(i)` while `a` and `b` are the number of admits and blocks. The following instructions

```
t=0:0.1:5000;
[m,a,b]=simblockswitch(10,0.1,120,t);
plot(t,m);
```

generated a simulation lasting 5,000 minutes. Here is a sample path of the first 100 minutes of that simulation:



The 5,000 minute full simulation produced `a=49658` admitted calls and `b=239` blocked calls. We can estimate the probability a call is blocked as

$$\hat{P}_b = \frac{b}{a+b} = 0.0048. \tag{1}$$

In the Markov Chains Supplement, we will learn that the exact blocking probability is given by the "Erlang-B formula." From the Erlang-B formula, we can calculate that the exact blocking probability is $P_b = 0.0057$.One reason our simulation underestimates the blocking probability is that in a 5,000 minute simulation, roughly the first 100 minutes are needed to load up the switch since the switch is idle when the simulation starts at time $t = 0$. However, this says that roughly the first two percent of the simulation time was unusual. Thus this would account for only part of the disparity. The rest of the gap between 0.0048 and 0.0057 is that a simulation that includes only 239 blocks is not all that likely to give a very accurate result for the blocking probability.