

LECTURE SUMMARY VII
Course: Communications Network I
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❖ **Highlights of this lecture:**

- Multidimensional Markov chains
- Truncation of independent single-class systems.
- Blocking probabilities for circuit switching systems.

❖ **Recap:**

(Lest you forgot...)

- Markov Chain:
A random sequence $X_0, X_1, \dots, X_n, \dots$ in which X_{n+1} depends on X_n but not on earlier values of the sequence X_0, X_1, \dots
- Birth-Death Process:
A processes in which state i can make transition only to state $i+1$ or $i-1$.
- Global balance equations:

$$p_j \sum P_{ji} = \sum p_i P_{ij} \quad j=0,1,2,\dots$$

These equations imply that at equilibrium, the frequency of transitions out of j equals that into j .

- Detailed balance equations:

$$p_j P_{ji} = p_i P_{ij} \quad i,j \geq 0$$

These equations imply that the frequency of transitions between two given states is equal in either direction.

- Global balance implies detailed balance, but the reverse is not true.
- Blocking probability:
The steady state probability that all circuits are busy simultaneously, in which case an arriving call is refused service.

[A] Multidimensional Markov chains.

This section concerns queuing systems with more than one classes of customers.

(a) Two class system with non-preferential treatment:

Features-

- A transmission line with m independent circuits each of capacity μ .
- Two types of sessions arriving with rates λ_1 and λ_2 .
- A session is blocked and is lost if all m circuits are busy upon its arrival.

Implications-

- The two sessions are indistinguishable for queuing purposes, as service rate μ is same for both.
- Each session can occupy up to all m circuits.

Markov chain model :

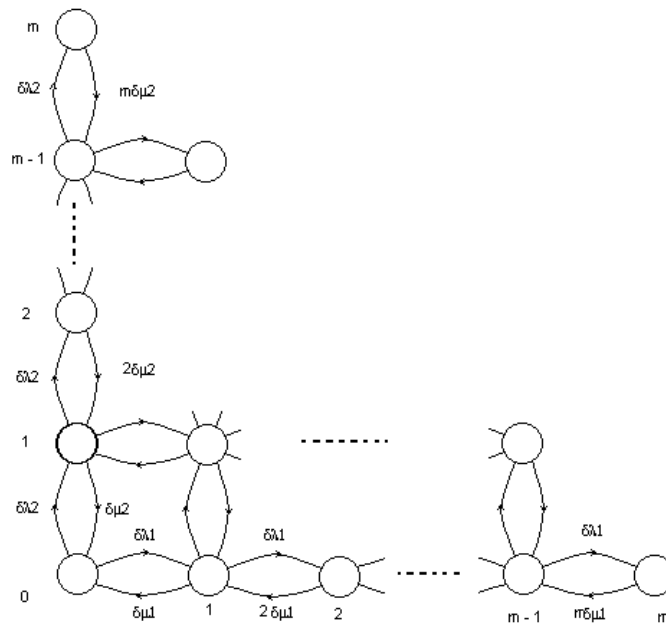


Figure 1: Markov chain for two-class system with non-preferential treatment

* λ_i represents the arrival rate for transitions from states i to $i+1$.

* μ_i represents the service rate for transitions from states i to $i-1$.

(b) Two class system with preferential treatment for one class.

Features-

- All queuing dynamics remain same as in (a) except the service rates.
- Class 1 can occupy all m circuits and so has a service rate of $\mu_1 = m\mu$.
- Class 2 can occupy only $k < m$ circuits and so it has a service rate of $\mu_2 = k\mu$

Implications-

- Maximum possible circuits that class 2 can occupy is k .
- Maximum possible circuits that class 1 can occupy is m , while they can occupy $m-k$ circuits in any case.
- The blocking probabilities for the two sessions are different.

Markov chain model:

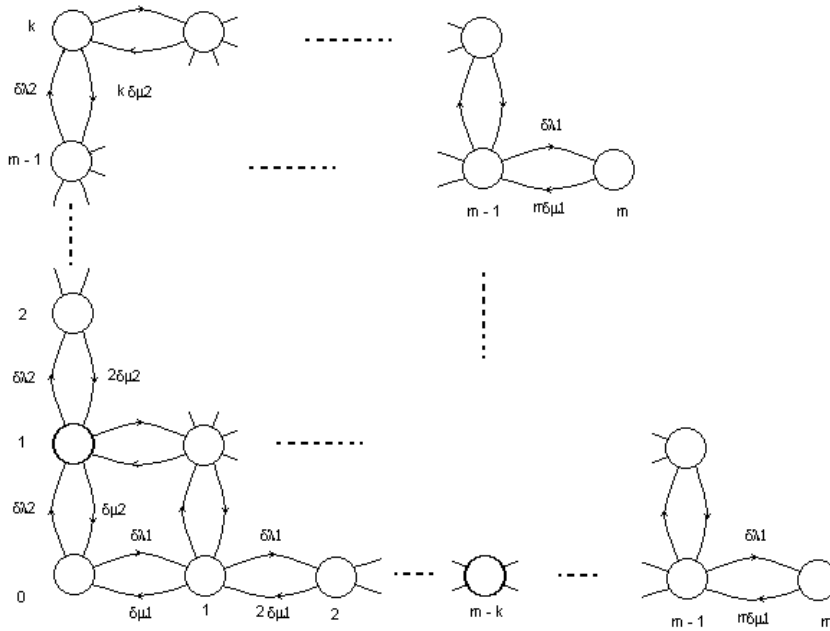


Figure 2 : Markov chain for two-class system with preferential treatment for one class

- * λ_i represents the arrival rate for transitions from states i to $i+1$
- * μ_i represents the service rate for transitions from states i to $i-1$

The stationary probability distribution is given as ,

$$P_{N_1 N_2}(n_1, n_2) = P_{N_1}(n_1) \cdot P_{N_2}(n_2)$$

Each state summarizes the past history in order to predict the future.

(c) K class systems

Multidimensional Markov chains with K customer classes are much harder to analyze. Here their states are of the form (n_1, n_2, \dots, n_k) where n_i is the number of customers of type i in the system.

However a closed-loop solution for the stationary distribution can be obtained in some special cases using following two properties...

(i) The detailed balance equations,

$$\lambda_i P(n_1, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_k) = \mu_i P(n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_k)$$

hold for all adjacent states,

$$(n_1, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_k) \text{ and } (n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_k)$$

where λ_i and μ_i are the arrival and the service rates of customers of type i.

(ii) The product form expression of stationary distribution,

$$P(n_1, n_2, \dots, n_k) = P_1(n_1) P_2(n_2) \dots P_k(n_k)$$

where for each i, $P_i(n_i)$ is an expression depending on the number n_i of the customer type i.

[B] Truncation of independent single-class system.

Truncation of states is a method used for analysis of multidimensional Markov chains. Here the analysis is restricted to single class systems.

Consider K independent M/M/1 queues for a multidimensional Markov chain that admits product form of solution.

The number of customers in the i th queue has the distribution,

$$P_i(n_i) = \rho_i^{n_i} (1 - \rho_i)$$

$$\text{where } \rho_i = \lambda_i / \mu_i$$

λ_i and μ_i are the corresponding arrival and service rates under the assumption of stable queue i.e. $\rho_i < 1$.

Noting that the K queues are independent and applying the product form,

$$P(n_1, n_2, \dots, n_k) = P_1(n_1) P_2(n_2) \dots P_k(n_k)$$

Now truncation is applied. The characteristics of state truncation are....

➤ Truncation gives a Markov chain having the same transition probabilities as before.

- The difference is that some states are eliminated while transitions between all other pairs of states along with their transition probabilities are left unchanged.
- The restriction on this procedure is that it should yield an irreducible Markov chain where all states communicate.
- In truncation an entire state has to be removed along with all transitions in and out of it.

The Markov chain models depicting the transitional probabilities before and after state truncation are shown below,

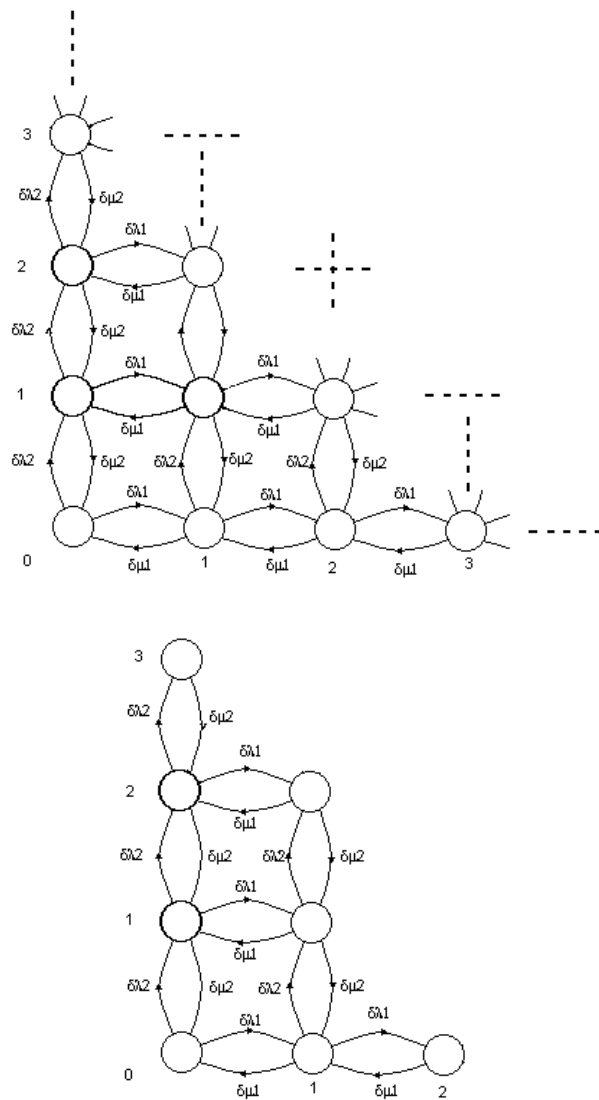


Figure4: Markov chain after truncation of the K independent M/M/1 queues for $K = 2$

* λ_i represents the arrival rate for transitions from states i to $i+1$.

* μ_i represents the service rate for transitions from states i to $i-1$.

The stationary distribution of this truncated system has the product form,

$$P(n_1, n_2, \dots, n_k) = (\rho_1^{n_1} \rho_2^{n_2} \dots \rho_k^{n_k}) / G \text{ -----(i)}$$

G is a normalization constant guaranteeing that $P(n_1, n_2, \dots, n_k)$ is a probability distribution.

$$\text{i.e. } G = \sum \rho_1^{n_1} \rho_2^{n_2} \dots \rho_k^{n_k} \text{ -----(ii)}$$

summation taken over $(n_1, n_2, \dots, n_k) \in S$ S is the set of truncated system.

Now, if a stationary probability distribution satisfies detailed balance then detailed balance is still okay after truncation.

Considering the detailed balance equations,

$$\lambda_i (P(n_1, n_2, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_k)) = \mu_i (P(n_1, n_2, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_k))$$

Substituting the probabilities from equation (i),

$$\lambda_i ((\rho_1^{n_1} \rho_2^{n_2} \dots \rho_{i-1}^{i-1} \rho_i^{n_i} \rho_{i+1}^{n_{i+1}} \dots \rho_k^{n_k}) / G) = \mu_i ((\rho_1^{n_1} \rho_2^{n_2} \dots \rho_{i-1}^{n_{i-1}} \rho_i^{n_i+1} \rho_{i+1}^{n_{i+1}} \dots \rho_k^{n_k}) / G)$$

This equation holds for an identity in view of the definition $\rho_i = \lambda_i / \mu_i$.

Hence detailed balance equations for truncated chain are satisfied. So it can be accepted as a unique state distribution.

* Calculation of G,

$$P(n_1, n_2, \dots, n_k) = \prod (1 - \rho_i) \rho_i^{n_i} \text{ product taken over } (i = 1 \text{ to } k)$$

and G is approximated by the number of samples belonging to the truncated space S.

* State distribution for k independent M/M/∞ queues is given by,

$$P(n_1, n_2, \dots, n_k) = ((\rho_1^{n_1} / n_1!) (\rho_2^{n_2} / n_2!) \dots (\rho_k^{n_k} / n_k)) / G$$

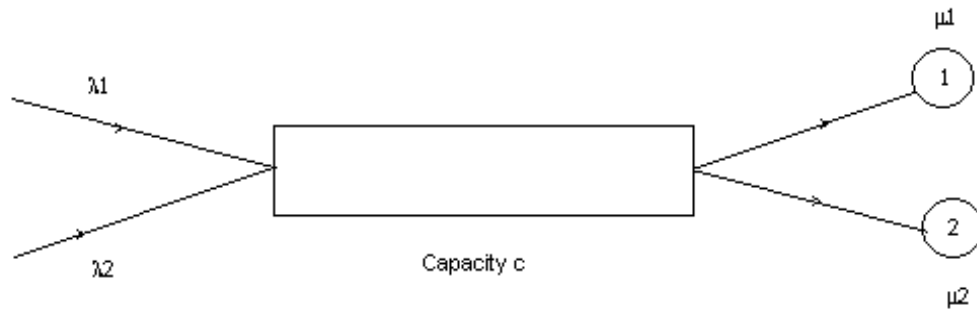
where ,

$$G = \sum (\rho_1^{n_1} / n_1!) (\rho_2^{n_2} / n_2!) \dots (\rho_k^{n_k} / n_k)$$

summation taken over the range $(n_1, n_2, \dots, n_k) \in S$

❖ Illustration :

Consider the following queuing system:



Features-

- Two classes of customers with arrival rates λ_1 and λ_2 .
- Two separate servers for each class with service rates μ_1 and μ_2 .
- A common waiting room with capacity c

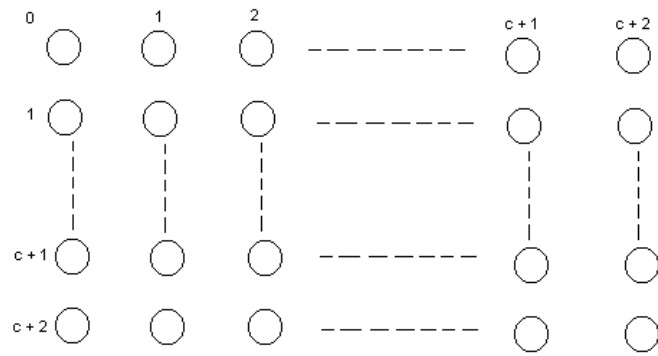
Implications-

- At any time maximum c+2 customers in the system.
- $P_{N1}(2) > 0, P_{N1}(c+1) > 0$
- $P_{N1N2}(0, c+2) = P_{N1N2}(1, c+2) = P_{N1N2}(2, c+2) = 0$ and so on...

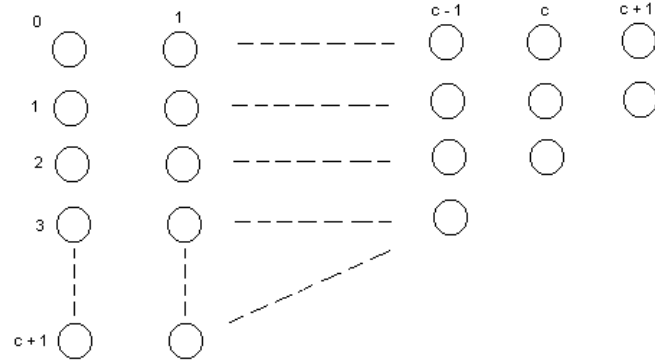
This is a clear candidate for state truncation, which can be carried out by eliminating the states with zero stationary probabilities.

The figure shows the Markov chain probability distribution for the cases before and after state truncation.

Before truncation :



After truncation :



This illustrates the procedure for truncation of states.

[C] Blocking Probabilities

The formulae for blocking probabilities for the system analyzed in case (b) are now obtained. The formulae are different from each other.

Kindly refer *Data Networks* (Bertsekas and Gallager 1985) for the listing of the formulae.

(..... The formulae are too weird for my limited WORD skills.....sorry.)

References:

- (i) D. Bertsekas and R. Gallager, *Data Networks*.
- (ii) R. Yates, *Communications Networks I class notes*.
- (iii) R. Yates and D. Goodman, *Probability and stochastic processes : A friendly introduction for electrical and computer engineers*.