# Wireless Communication Technologies

16:332:559 (Advanced Topics in Communications)
Lecture #21 and #22 (April 15, April 17, 2002)
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## Multiuser Optimum and Sub-optimum Detectors

The main performance measure of interest in digital communications in general, and in multi-user detection in particular, is the bit-error-rate $P_k(\sigma)$ as discussed in previous lectures. In addition, there are several performance measures derived from the bit-error-rate that are useful in the analysis, design and understanding of various detectors. Such two performance measures viz.asymptotic multi-user efficiency and near-far resistance were introduced in previous lecture.
Those performance measures are repeated here for the sake of convenience.

*Asymptotic Multi-user Efficiency:*

$$\eta_k = \sup\left\{ 0 \le r \le 1; \lim_{\sigma \to 0} \frac{P_k(\sigma)}{Q\left(\frac{\sqrt{r}A_k}{\sigma}\right)} = 0 \right\}$$

$$= \frac{2}{A^2_k} \lim_{\sigma \to 0} \sigma^2 \log\left[\frac{1}{P_k(\sigma)}\right]$$

*Near-Far resistance:*

$$\eta_k = \inf_{A_j > 0,\, j \ne k} \eta_k$$

These metrics measure the robustness of the system to interfering powers.
The following section analyses robustness of conventional receiver for multi-user detection by calculating its multi-user efficiency and near-far resistance.

### Conventional Receiver (Matched Filter) in the CDMA channel:

Let us consider the two-user (K=2) synchronous CDMA model.
As derived in the previous lecture, when eye is closed $A_1 \le A_2|\rho|$, the matched-filter error probability is given by

$$P^c_1 = \frac{1}{2}Q\left(\frac{A_1 - A_2|\rho|}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1 + A_2|\rho|}{\sigma}\right)$$ ,which does not vanish as σ ->0

$\therefore \eta^{c}_{1} = 0$

If eye is open $A_1 \geq A_2 |\rho|$ then,

$$\lim_{\sigma \to 0} \frac{P^{c}_{1}(\sigma)}{Q\left(\frac{\sqrt{r}A_1}{\sigma}\right)} = \lim_{\sigma \to 0} \frac{\frac{1}{2}Q\left(\frac{A_1 - A_2|\rho|}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1 + A_2|\rho|}{\sigma}\right)}{Q\left(\frac{\sqrt{r}A_1}{\sigma}\right)}$$

Using L'Hospital's rule, we get,

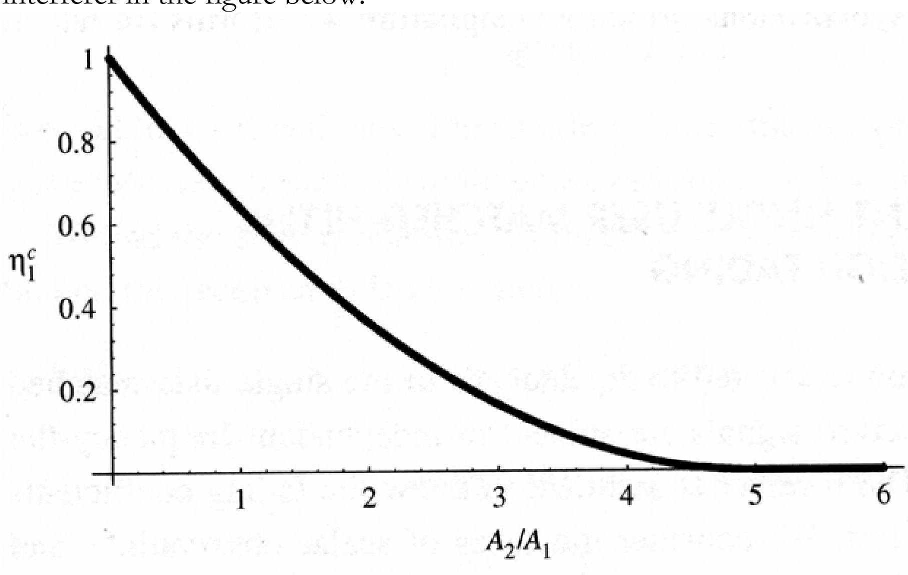
$$= \begin{cases} 0, \sqrt{r}A_1 < A_1 - A_2|\rho| \\ +\infty, \sqrt{r}A_1 > A_1 - A_2|\rho| \end{cases}$$

$$\eta^{c}_{1} = \left(1 - \frac{A_2}{A_1}|\rho|\right)^2 \qquad [1]$$

Putting together the asymptotic multi-user efficiency in both regions (eye-open and eye-close) can be written as,

$$\eta^{c}_{1} = \max{}^2\left\{0, \left(1 - \frac{A_2}{A_1}|\rho|\right)\right\} \qquad [2]$$

The asymptotic multi-user efficiency is plotted as a function of the relative amplitude of the interferer in the figure below.



Proceeding analogously in the K-user synchronous and asynchronous channels we get,

$$\eta^{c}_{k} = \max{}^2 \left\{0, 1 - \sum_{j \neq k} \frac{A_j}{A_k} \left|\rho_{jk}\right|\right\} \qquad \text{(Synchronous)} \qquad [3]$$

$$\eta^{c}_{k} = \max{}^2 \left\{0, 1 - \sum_{j \neq k} \frac{A_j}{A_k} (\left|\rho_{jk}\right| + \left|\rho_{kj}\right|)\right\} \qquad \text{(Asynchronous)} \qquad [4]$$

The asymptotic efficiency of the conventional detector can be seen as a normalized measure of the eye-opening.

Minimizing eqn. 3 or 4 over $\left\{A_j, j \neq k\right\}$ we see that the near-far resistance of the *k*th user is equal to 0 unless $\rho_{jk} = \rho_{kj}$ for all $j \neq k$.

In other words, the *k*th user signature waveform must be orthogonal to each of the partially overlapping waveforms of every interferer. Because this condition cannot be satisfied for all offsets in an asynchronous channel, we conclude that the conventional receiver is not near-far resistant, except in the trivial case of synchronous orthogonal signature waveforms (in that case it is optimal)

After analyzing the performance of conventional detector in multi-user system, the next step is to find the optimum strategy for multi-user detection.

**Optimum Detector** (Verdu, 1983):

The conventional single-user matched filter receiver requires no knowledge beyond the signature waveforms and timing of the users it wants to demodulate. In the following derivation of an optimum receiver, it is assumed that the receiver not only knows the signature waveform and timing of every active user, but it also knows (or can estimate) the received amplitudes of all users and the noise level.

Consider a synchronous channel

$$y(t) = \sum_{k=1}^{K} A_k b_k S_k(t) + \sigma n(t) \qquad t \in [0, T] \qquad [5]$$

Optimum decision rule is MAP (maximum a posteriori rule). However, the optimum detection can be viewed as *"individually optimum detection"* as well as *"jointly optimum detection"* as explained below:

Individual optimum detection strategy that maximizes the a posteriori probability is written as:

$\max_{b_k} P[b_k / y(t)]$ for k=1,2…K

While Jointly Optimum detection strategy that maximizes the joint a posteriori probability is written as:

$$\max_{b_1,..b_k} P[b_1, b_2..b_k / y(t)] \quad \text{for k=1,2...K}$$

Now consider the case of two-user synchronous channel, so that the equation 5 becomes

$$y(t) = A_1 b_1 s_1(t) + A_2 b_2 s_2(t) + \sigma n(t) \qquad t \in [0, T]$$

The minimum probability of error decision for user 1 is obtained by selecting the value of $b_1 \in \{-1, +1\}$ that maximizes the a posteriori probability

$$\max_{b_1} P[b_1 / y(t)] \qquad [6]$$

The other optimum detection problem by requiring that the receiver selects the pair ($b_1$, $b_2$) that maximizes the joint a posteriori probability

$$\max_{b_1, b_2} P[b_1, b_2 / y(t)] \qquad [7]$$

Equation 6 can be written in terms of 7 as,

$$P[b_1 / y(t)] = P[(b_1, +1) / y(t)] + P[(b_1, -1) / y(t)] \qquad [8]$$

Let us take an example where the noise realization is such that the a posteriori probabilities take the following values:

$$P[(+1,+1)/y(t)] = 0.26$$
$$P[(-1,+1)/y(t)] = 0.26$$
$$P[(+1,-1)/y(t)] = 0.27$$
$$P[(-1,-1)/y(t)] = 0.21$$

From above equations it is clear that the jointly optimum decisions are ($b_1$, $b_2$)=(+1, -1), whereas the individually optimum decisions are ($b_1$, $b_2$)=(+1, +1).

The reason for the "apparent discrepancy" is that the $b_1$ and $b_2$ are not independent when conditioned on the observed waveform and hence the jointly optimum and individually optimum decisions need not coincide. However, when signal-to-noise ratio is sufficiently high both types of decisions agree with high probability.

Let us now consider the K-user basic synchronous CDMA channel:

$$y(t) = \sum_{k=1}^{K} A_k b_k S_k(t) + \sigma n(t) \qquad t \in [0, T]$$

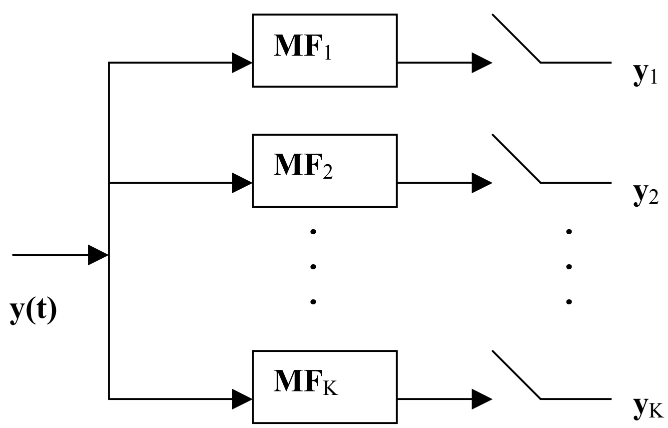
The problem is jointly optimum demodulation of

$\mathbf{b}=[b_1, b_2 \ldots b_K]^T$

For the case of n (t) being AWGN, the optimum receiver which is the ML (maximum likelihood) receiver is also the minimum-probability-of-error receiver.

i.e. $\hat{\underline{b}}_{ML} = \arg\max_{b} P[y(t), t \in [0,T] / \underline{b}]$

It can be shown that the sufficient statistic for ML detection is **y=**$[y_1, y2\ldots y_K]^T$ which is the column vector of matched-filter outputs as shown below.




$$y_k = \int_0^T y(t) s_k(t) dt$$

$y_k = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_K$ and **n=**$[n_1, n2\ldots n_K]^T$

**n** is jointly gaussian vector with E (**n**)=0 and E $(n_j n_l) = \sigma^2 \varrho_{jl}$

So we have,

**y=RAb** + **n**

where,

**R** is the normalized cross-correlation matrix whose diagonal elements are equal to 1 and whose (i, j) element is equal to the cross-correlation $\varrho_{ij}$

**A** is the K*K diagonal matrix of received amplitudes

**A=diag** $\{A_1,\ldots A_K\}$

and the Covariance matrix is given as Cov (**n**)=$\sigma^2$**R**

$$p\,[\mathbf{y}/\mathbf{b}] = \exp\left\{ \frac{-\frac{1}{2}\left\{(\underline{y} - RA\underline{b})^T (\sigma^2 R)^{-1} (\underline{y} - RA\underline{b})\right\}}{\sqrt{2\pi\sigma^2 |R|}} \right\}$$

$\therefore$ ML rule says

$$\hat{\underline{b}}_{ML} = \arg\max_{\underline{b}} \Omega(\underline{b})]$$

Where

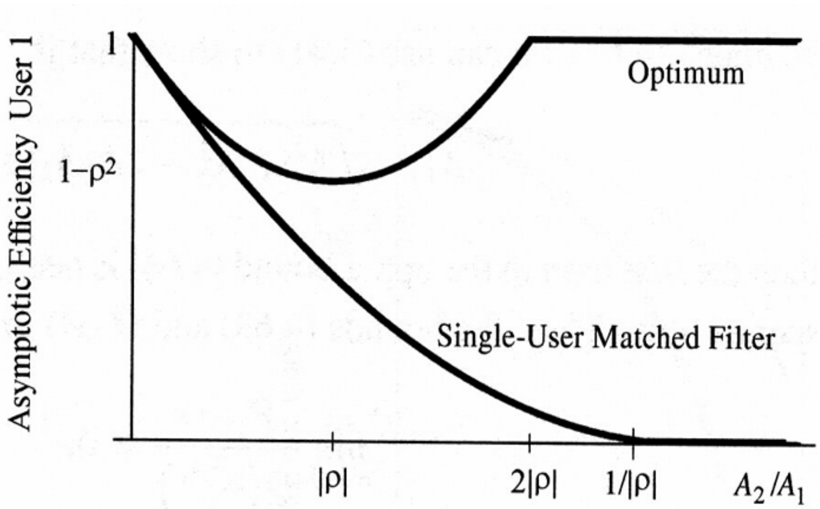
$$\Omega(\underline{b}) = 2\underline{b}^T A\,\underline{y} - \underline{b}^T H\,\underline{b}, H = ARA \qquad [9]$$

Equation 9 reveals that the dependence of the likelihood function on the received signals is through the vector of matched filter outputs **y,** which is therefore sufficient statistic for demodulating the transmitted data.

The maximization of 9 is a combinatorial optimization problem, which can be solved by exhaustive search, namely, compute the function for every possible argument and select the one that maximizes the function. The computational complexity of any detector can be quantified by its time complexity per bit, that is, the number of operations required by the detector to demodulate the transmitted information divided by the total number of demodulated bits. The time complexity per bit for the selection of optimum **b** is O ($2^K$/K) (NP complete)

For K=2,the optimum receivers' asymptotic multi-user efficiency is given by,

$$\eta_1^{OPT} = \min\left\{1, 1 + \frac{A_2^{\,2}}{A_1^{\,2}} - 2|\rho|\frac{A_2}{A_1}\right\} \qquad [10]$$



The above figure shows the asymptotic efficiency of user 1 for both optimum and conventional receiver. It can be seen that for optimum receiver asymptotic efficiency is not monotonic in $A_2/A_1$.Actually, if

$$\frac{A_2}{A_1} \geq 2|\rho|$$

then $\eta_1 = 1$.

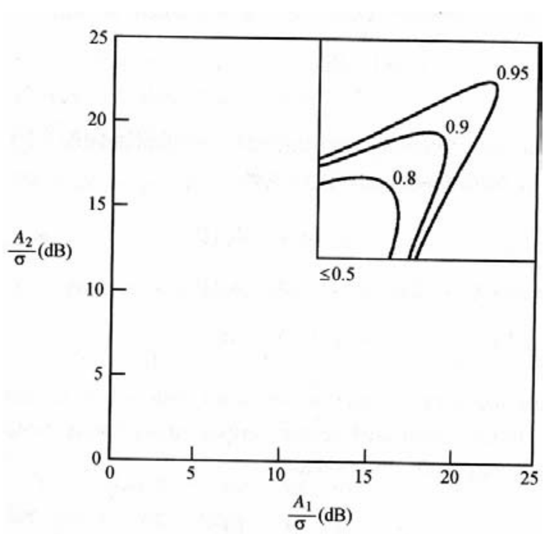
Therefore, as long as the energy of user 2 exceeds the threshold given by above equation the asymptotic bit-error-rate of user 1 is equivalent to the single-user case where user 2 is not active The explanation of this behavior of the optimum receiver is that if the interfering user is sufficiently powerful, then the primary source of errors committed in the optimum demodulation of user 1 is the background Gaussian noise, rather than the randomness of the information carried by the interfering signal. This fact could be explained using the successive decoding technique.

The near-far resistance is obtained by minimizing equation 10 over $\frac{A_2}{A_1} \geq 0$

The least favorable relative amplitude of user 2 is which yields the near-far resistance for either user:

$$\bar{\eta}_k = 1 - \rho^2$$

The figure below shows the two-user power-tradeoff region so that the optimum bit-error-rate of both the users is not higher than $3\times10^{-5}$, for $|\varrho|$=0.8,0.9 and 0.95.If we compare this figure with the one for conventional receiver, we can conclude that the permissible signal-to-noise ratios are indistinguishable as long as the cross-correlations satisfies $\rho \leq 0.5$. Also for high cross-correlations values, equal powers for users are detrimental. The reason is that if both signature waveforms are very much alike, then the similar amplitudes complicate the task of the optimum receiver.



The complexity of optimum multi-user detector requires one to come-up with other multi-user detectors that exhibit good performance and complexity tradeoffs. The next section considers one such sub-optimum receiver.

## Sub-Optimum Receivers:

### *Decorrelating Detector:*

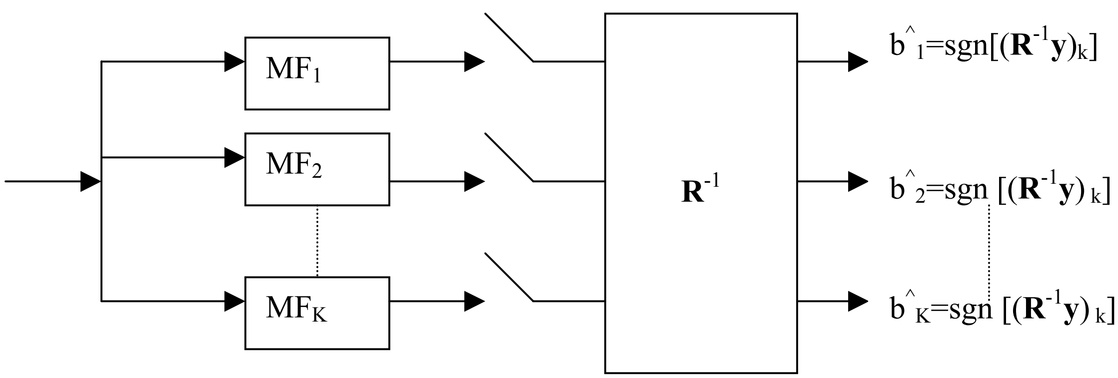
Consider the output vector of the bank of K matched filter outputs:
$\mathbf{y}=\mathbf{RAb}+\mathbf{n}$

where $\mathbf{n}$ is the gaussian random vector with zero mean and covariance vector $\sigma^2\mathbf{R}$

Let us assume that the cross-correlation matrix $\mathbf{R}$ is invertible. If we premultiply the vector of matched filter outputs by $\mathbf{R}^{-1}$, then

$$\mathbf{R}^{-1}\mathbf{y}=\mathbf{Ab}+\mathbf{R}^{-1}\mathbf{n} \qquad [11]$$

The *k*th component of equation 11 is free from interference caused by any other users, that is, it is independent of all $\{b_j\}$, $j \neq k$ ($\mathbf{A}$ is a diagonal matrix). The only source of interference is the background noise. That is why the detector that performs 11 is called decorrelating detector.




In the absence of noise, that is, $\sigma=0$,we have $\hat{b}_K=\text{sgn}\,[(\mathbf{R}^{-1}\mathbf{y})_k]=\text{sgn}\,[(\mathbf{Ab})_k]=\mathbf{A}\text{sgn}\,(b_k)=b_k$
So we see that in the absence of noise this detector gives error-free performance, unlike conventional detector. Also the cross-correlation matrix $\mathbf{R}$ is invertible if the signature waveforms are linearly independent.

From the implementation point of views, the desirable features of this multiuser detector are:
1.It does not require knowledge of the received amplitudes
2.It can readily be decentralized, in the sense that the demodulation of each user can be implemented completely independently

The *k*th output of the linear transformation $\mathbf{R}^{-1}$ is

$$(R^{-1}\underline{y})_k = \sum_{j=1}^{K} R_{kj}^{\ +} y_j = \sum_{j=1}^{K} R_{kj}^{\ +} < y, s_j >$$
$$= < y, \sum_{j=1}^{K} R_{kj}^{\ +} s_j > = < y, \tilde{s}_k >$$

Where $(\mathbf{R}^{-1})_{kj}$ is denoted as $\mathbf{R}_{kj}^{\ +}$ and $\tilde{s}_k = \sum_{j=1}^{K} R_{kj}^{\ +} s_j$

Therefore the decorrelator for $k$th user can be implemented as $\operatorname{sgn}\left\{\int_0^T y(t)\tilde{s}_k(t)dt\right\}$ which can be viewed as the implementation of modified matched filter that is matched to $\tilde{s}_k(t)$.

The unit inner product of $\tilde{s}_k(t)$ with its corresponding signature waveform will yield

$$< \tilde{s}_k, s_k > = \int_0^T \sum_{j=1}^{K} R^{+}{}_{jk} s_j(t) s_k(t) dt = [R^{-1}R]_{kk} = 1 \qquad [12]$$

It should also be noted that the decision statistic of the decorrelating detector contains no trace of the signals modulated by the interfering users.
Since for an vector $(a_1,...a_K) \in R^K$,

$$\int_0^T \left[\sum_{i\neq k} a_i s_i(t)\right]\tilde{s}_k(t)dt = \int_0^T \left[\sum_{i\neq k} a_i s_i(t)\right]\left[\sum_{j=1}^{K} R^{+}{}_{jk} s_j(t)\right]dt$$
$$= \sum_{i\neq k} a_i [R^{-1}R]_{ik}$$
$$= 0$$

In other words one can say that the decorrelating linear transformation is the projection of the signal of the desired user on the orthogonal space to the space spanned by the interfering signals, and, thus its bit-error-rate is invariant to the amplitudes of the interfering signals.

The output of the filter matched to $\tilde{s}_k(t)$ (modified matched filter) has only two components: one due to the signal of the user k, which is equal to $A_k b_k$ (from eqn12), and the other due to the background noise, which is Gaussian with zero mean and the variance equal to the $kk$ component of the covariance matrix

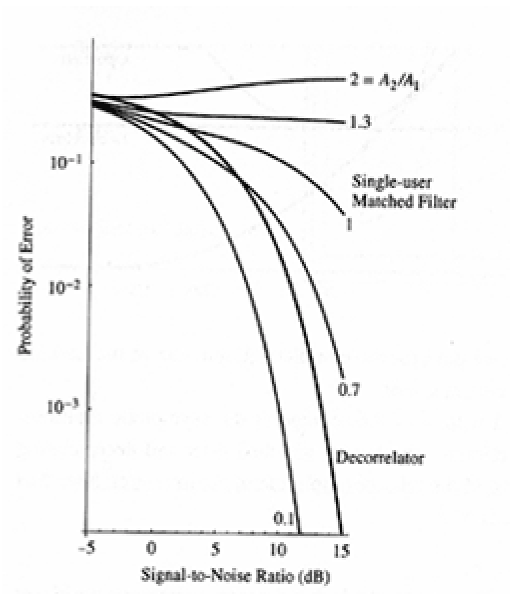
$$E\,[(\mathbf{R}^{-1}\mathbf{n})(\mathbf{R}^{-1}\mathbf{n})^{T}] = E\,[\mathbf{R}^{-1}\mathbf{n}\,\mathbf{n}^{T}\,\mathbf{R}^{-1}]$$
$$= \sigma^2\,\mathbf{R}^{-1}\,\mathbf{R}\,\mathbf{R}^{-1}$$
$$= \sigma^2\,\mathbf{R}^{-1}$$

Consequently, the $k$th user bit-error-rate is

$$P^d{}_k(\sigma) = Q\left(\frac{A_k}{\sigma\sqrt{R_{kk}{}^+}}\right) \qquad [13]$$

If the *k*th user is orthogonal to the other users, then $R^+_{kk}=1$ and the decorrelator coincides with the single-user matched filter.

The figure below shows the bit-error-rates of decorrelator and single-user matched filter with two users and $\varrho=0.75$




It is noted that if the interfering amplitude is small enough, the single-user matched filter detector is preferable to the decorrelator. This is because even though the components in the respective decision statistics due to the desired user are identical in both cases, the component due to the noise has variance $\sigma^2$ for the single-user matched filter detector versus variance $\sigma^2/(1-\varrho^2)$ for the decorrelating detector. Thus, the price paid for the complete elimination of multi-access interference is "noise enhancement".

*Asymptotic Mutliuser efficiency and Near-Far resistance of Decorrelator:*

From equation 13,we see that the SNR required to achieve equivalent bit-error-rate as of decorrelator is

$$\frac{A^2{}_k}{\sigma^2 R^+{}_{kk}}$$

and so the multi-user efficiency is equal to

$$\eta_k{}^d = \frac{1}{R_{kk}{}^+}$$

As we see that the multi-user efficiency doesn't depend on either the noise level or the interfering amplitudes, it is equal to asymptotic multi-user efficiency and near-far resistance, that is,

$$\bar{\eta}_k^{\,d} = \frac{1}{R_{kk}^{+}}$$

The decorrelating detector achieves the maximum near-far resistance.

It can be shown that the decorrelator is optimum detector when received amplitudes are unknown. If the received amplitudes are unknown then it is natural to consider joint maximum-likelihood estimation of amplitudes and transmitted bits. Because the noise is white and gaussain, the most likely bits and amplitudes are those that best explain the received waveform in a mean-square sense, that is, the arguments that achieve

$$\min_{b\in\{-1,1\}^K} \min_{A_k \geq 0, k=1,..K} \int_0^T \left[ y(t) - \sum_{k=1}^{K} A_k b_k s_k(t) \right]^2 dt \qquad [14]$$

If we let $c_k = A_k b_k$, then it can be shown that the minimization of equation 14 is equivalent to the maximization of

$$\max_{c\in R^K} 2\,\underline{c}^T \underline{y} - \underline{c}^T \underline{R}\, \underline{c} \qquad [15]$$

Taking gradient of 15 w.r.t **c** and set to zero we get,

$\mathbf{R}\mathbf{c}^* = \mathbf{y}$, that is

$\mathbf{c}^* = \mathbf{R}^{-1}\mathbf{y}$

Then the most likely bits and amplitudes are

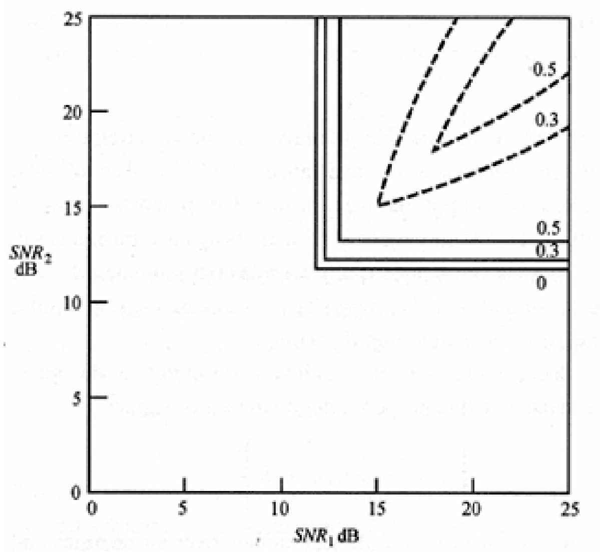
$$\hat{b}_k = \mathrm{sgn}(c^*{}_k) = \mathrm{sgn}[(\underline{R}^{-1}\underline{y})_k]$$

and

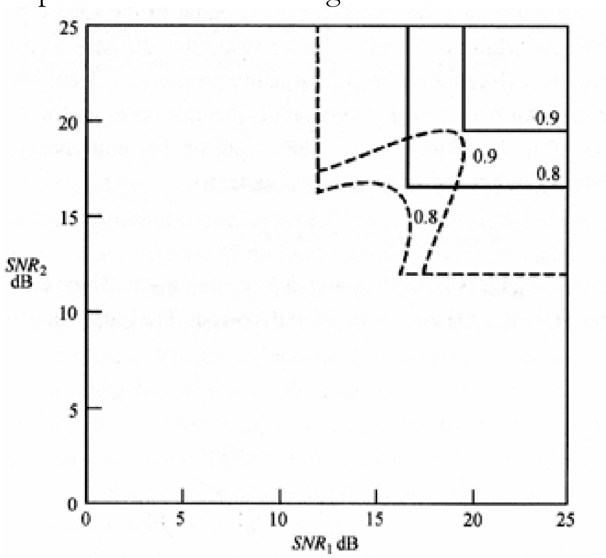
$$\hat{A}_k = \left|c^*{}_k\right|$$

Therefore, the decorrelating detector is seen to give the best joint estimate of the transmitted bits and amplitudes in the absence of any prior knowledge about the received amplitudes.

The figure below compares the power-tradeoff regions (for $BER \leq 3\times10^{-5}\, and \left|\rho\right| = 0, 0.3, 0.5$) for decorrelating detector and the single-user matched filter (dashed). Since the decorrelating detector bit-error-rate is independent of the amplitude of the interferers, the power tradeoff region is always a quadrant as shown

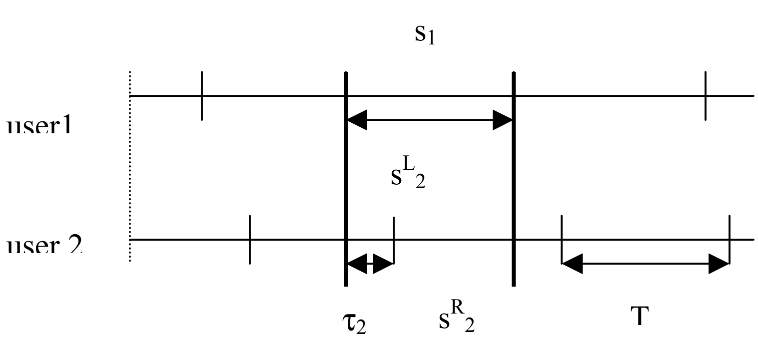



This figure below compares the power-tradeoff regions (for $BER \leq 3\times10^{-5}\, and\, |\rho| = 0.8, 0.9$) for decorrelating detector and the optimum detector (dashed). It can be seen that the two-user optimum detector offers marginal gains with respect to the decorrelating detector when both amplitudes are equal.




**Asynchronous decorrelator detector (K=2):**

The situation in the demodulation of user 1 in asynchronous case with two-users is as depicted below:



So user 1 is effectively interfered by 2 interferers (left-bit, right-bit partial signature correlations).
This 2 user asynchronous system then can be considered as a three-user synchronous channel where the interferers have unit-energy signature waveforms:

$$s_2{}^L(t) = \frac{1}{\sqrt{\theta_2}} s_2(t+T-\tau_2), if\, 0 \le t \le \tau_2$$

$$= 0, if\, \tau_2 < t \le T$$

$$s^R{}_2 = 0, if\, \tau_2 < t \le T$$

$$= \frac{1}{\sqrt{1-\theta_2}} s_2(t-\tau_2), if\, \tau_2 \le t \le T$$

Where $\theta_2$ is the partial energy of the interfering signal over the left over-lapping interval.
The crosscorrelation matrix of the "three-user synchronous" channel is:

$$\mathbf{R}=\begin{bmatrix} 1 & \rho_{21}/\sqrt{\theta_2} & \rho_{12}/\sqrt{1-\theta_2} \\ \rho_{21}/\sqrt{\theta_2} & 1 & 0 \\ \rho_{12}/\sqrt{1-\theta_2} & 0 & 1 \end{bmatrix}$$

The first row of $\mathbf{R}^{-1}$ is a constant times the vector $\begin{bmatrix} 1 & -\rho_{21}/\sqrt{\theta_2} & -\rho_{12}/\sqrt{1-\theta_2} \end{bmatrix}$
Therefore, the two-user one-shot decorrelator described above, subtracts from the matched filter output of the desired user the weighted outputs of the partial correlators. The above concept of one-shot decorrelation can be extended to any number of users.

## Approximate Decorrelator (Mandayam-Verdu):

If the normalized crosscorrelations among all the signature waveforms are very small, **R** is strongly diagonal.
That is
$\mathbf{R}^{-1}=(\mathbf{I}+\delta\mathbf{M})^{-1}=\mathbf{I} - \delta\mathbf{M} + o(\delta)$

So the result is that for the $k$th user the approximation results in a modified matched filter for the synchronous as:

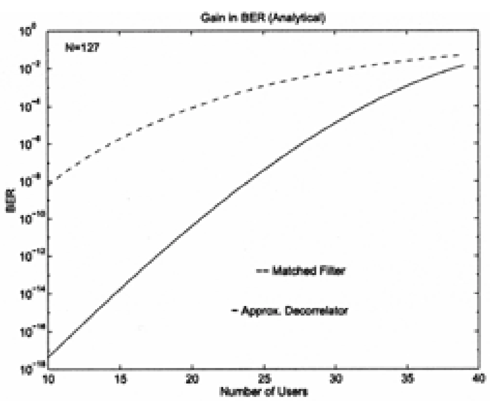
$$\tilde{s}_k(t) = s_k(t) - \sum_{j \ne k} \rho_{jk} s_j(t) \qquad [16]$$

For the asynchronous case, the $k$th user can be approximated by a filter matched to:

$$\tilde{s}_k(t) = s_k(t) - \sum_{j \neq k} \rho_{jk} s_j(t - \tau_j) - \sum_{j \neq k} \rho_{kj} s_j(t - \tau_j + T)$$

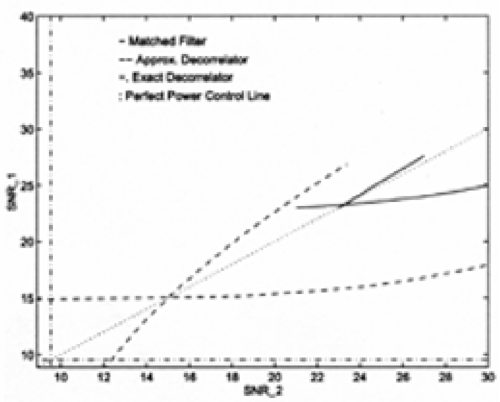
Whenever the crosscorrelations are not known in advance and the detector coefficients have to be computed on-line, the approximation in equation 15 has the advantage that it does not need any processing of the crosscorrelations supplied by the crosscorrelators of the replicas of the signature waveforms. The reduced complexity of the approximate decorrelator and performance gains over the conventional matched filter makes it a viable alternative for implementation in practical CDMA systems, in particular in those where the signature waveforms span many symbol intervals. The near-far resistance of approximate decorrelator is zero, but its bit-error-rate performance has been shown to be quite superior to that of the conventional matched filter. In fact it can be proved that as long as the load factor $K/N < 1/3$ the bit-error-rate performance of approximate decorrelator is better than the single-user matched filter [3].

The figures shown below compare the approximate decorrelator and single-user matched filter for random signature sequences and perfect power control.



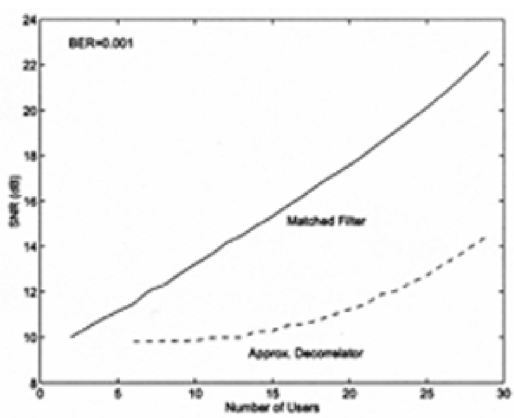

BER Gain

$P_k = P_1 = P$, and $P/\sigma_N^2 = 20dB$




Power Trade-off Regions

$N = 127$, $K = 30$, and BER $\leq 0.001$




Required SNR (Perfect Power Control)

$N = 127$, BER $= 0.001$

**References:**

1.**Sergio, Verdu** "Multiuser Detection", Cambridge University Press, 1998
2.**Narayan Mandayam and Sergio Verdu** "Analysis of an Approximate Decorrelating Detector", Wireless Personal Communications, vol. 6, No. 1/2, pp. 97-111, January, 1998
3.**Narayan Mandayam**, "Lecture Notes 21 and 22,Wireless Communication Technologies 16:332:559", RUTGERS University, April, 2002