Communications Engineering
Course No: 16:332:421 - (Fall 2004)

Solutions to Quiz 2

Instructions: Answer all questions. Maximum Marks: 25. The points for each question are listed below in parentheses. You are allowed a total time of 1 hour and 20 minutes.

1. The analog signal is sampled at $f_s = 8 \text{ KHz}$. (8)
   Each sample is quantized with $L = 64$ levels of representation. Therefore the number of bits $R$ required to represent each sample is
   
   $$ R = \log_2 L = 6 \text{ bits} $$

   The total bit rate after sampling and quantization is $f_s \times R \text{ Kbps}$.

   The minimum transmission bandwidth required $W$ is given as $W = \frac{1}{2T}$, where $T$ is the symbol duration of the $M$-ary PAM system.

   (a) $M = 2$
   For $M = 2$ amplitude levels, each pulse can represent $\log_2 M = \log_2 2 = 1 \text{ bit}$. Therefore,
   
   $$ T = \frac{1}{f_s R \log_2 M} = \frac{1}{f_s R} $$

   $$ \Rightarrow W = f_s R / 2 = 48 / 2 \text{ KHz} = 24 \text{ KHz} $$

   (b) $M = 4$
   For $M = 4$ amplitude levels, each pulse can represent $\log_2 M = \log_2 4 = 2 \text{ bits}$. Therefore,
   
   $$ T = \frac{1}{f_s R \log_2 M} = \frac{1}{f_s R} \times 2 $$

   $$ \Rightarrow W = f_s R / 4 = 48 / 4 \text{ KHz} = 12 \text{ KHz} $$

2. The channel bandwidth is given to be $B = 60 \text{ KHz}$ and the bit rate is $R_b = 100 \text{ Kbps}$. (4)
   The bit duration is therefore given as $T_b = \frac{1}{R_b} = 10 \mu\text{sec}$.

   The signal bandwidth can be found as $W = \frac{1}{2T_b} = 50 \text{ KHz}$

   Therefore, the raised cosine pulse should be designed such that its rolloff factor $\alpha$ satisfies

   $$ B = W(1 + \alpha) $$

   $$ \Rightarrow \alpha = 0.2 $$
3. Consider the set of signals \( \{ s_i(t) \}_{i=1}^{4} \), where the signal \( s_i(t) \) is of the form

\[
s_i(t) = \begin{cases} 
\sqrt{\frac{2E}{T}} \cos(2\pi \frac{t}{T} + i \frac{\pi}{4}), & 0 \leq t \leq T \\
0, & \text{otherwise}
\end{cases}
\]

Observe that using the cosine formula \( \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \), we can write each of the above signals as

\[
s_i(t) = \begin{cases} 
\sqrt{\frac{2E}{T}} [\cos(2\pi \frac{t}{T}) \cos(i \frac{\pi}{4}) - \sin(2\pi \frac{t}{T}) \sin(i \frac{\pi}{4})], & 0 \leq t \leq T \\
0, & \text{otherwise}
\end{cases}
\]

Therefore each signal can be written as a weighted sum of the two functions \( \cos(2\pi \frac{t}{T}) \) and \( \sin(2\pi \frac{t}{T}) \). Do these two functions make an orthonormal basis?

They do if we choose \( \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi \frac{t}{T}) \) and \( \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi \frac{t}{T}) \), since we can easily verify that

\[
\int_0^T \phi_i(t) \phi_j(t) dt = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j
\end{cases}
\]

Therefore, each of the signals can now be written as

\[
s_i(t) = \begin{cases} 
\sqrt{E} \cos(i \frac{\pi}{4}) \ \phi_1(t) - \sqrt{E} \sin(i \frac{\pi}{4}) \ \phi_2(t), & 0 \leq t \leq T \\
0, & \text{otherwise}
\end{cases}
\]

Therefore the coefficients in the expansion are

\[
s_{11} = \sqrt{E} \cos(\frac{\pi}{4}) = \sqrt{E}/2, \quad s_{12} = -\sqrt{E} \sin(\frac{\pi}{4}) = -\sqrt{E}/2
\]

\[
s_{21} = \sqrt{E} \cos(\frac{2\pi}{4}) = 0, \quad s_{22} = -\sqrt{E} \sin(\frac{2\pi}{4}) = -\sqrt{E}
\]

\[
s_{31} = \sqrt{E} \cos(\frac{3\pi}{4}) = -\sqrt{E}/2, \quad s_{32} = -\sqrt{E} \sin(\frac{3\pi}{4}) = -\sqrt{E}/2
\]

\[
s_{41} = \sqrt{E} \cos(\frac{4\pi}{4}) = -\sqrt{E}, \quad s_{42} = -\sqrt{E} \sin(\frac{4\pi}{4}) = 0
\]

4. Consider a set of orthonormal basis functions \( \{ \phi_j(t) \}_{j=1}^{N} \).

Let \( w(t) \) be an AWGN process of zero mean and p.s.d. 1.

We will show that the sequence \( \{ w_j \}_{j=1}^{N} \) are i.i.d. Gaussian random variables, where

\[
w_j = \int_0^T w(t) \phi_j(t) dt, \quad j = 1, \ldots, N.
\]

Since \( w(t) \) is a Gaussian process, it follows that \( w_j \) is a Gaussian random variable. Further, \( E[w_j] = 0 \), since \( w(t) \) is zero mean.
Consider the covariance function

\[ \text{Cov}(w_j, w_k) = E[w_j w_k] = E[\int_0^T w(t)\phi_j(t)dt \quad \int_0^T w(t)\phi_k(t)dt] \]

Rearranging the integrals ⇒

\[ \text{Cov}(w_j, w_k) = E[\int_0^T \int_0^T w(t)\phi_j(t)w(u)\phi_k(u)dtdu] \]

Taking the expectation inside the integral ⇒

\[ \text{Cov}(w_j, w_k) = \int_0^T \int_0^T \phi_j(t)\phi_k(u)E[w(t)w(u)]dtdu \]

But \( E[w(t)w(u)] = \delta(t-u) \) ⇒

\[ \text{Cov}(w_j, w_k) = \int_0^T \phi_j(t)\phi_k(t)dt = 0 \]

⇒ \( w_j \) and \( w_k \) are uncorrelated.

When \( j = k \), \( \text{Cov}(w_j, w_j) = \text{Var}(w_j) = 1 \) ⇒ the random variables \( w_j \) have the same variance as well.

Therefore, the sequence \( \{w_j\}_{j=1}^N \) are uncorrelated and identically distributed. Since they are Gaussian, it follows that they are also independent.