1. A bit 1 is represented by a pulse of height $A$ for a duration of 1 second and a bit 0 is represented by sending no pulse for a duration of 1 second. The signals are transmitted over an AWGN channel with zero mean and power spectral density $1/2$. Let $y$ denote the output of the integrator in Figure 1.

![Figure 1: Receiver for the PCM System with On-off Keying](image)

(a) For equiprobable bit-transmission, $p_0 = p_1 = 1/2$. To find the optimum threshold $\lambda$ that minimizes the probability of error, we need to solve the following equation

$$\frac{p_0}{p_1} = 1 = \frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda_{opt}|0)} \tag{1}$$

Let us first find the density functions $f_Y(y|1)$ and $f_Y(y|0)$

When a 1 is transmitted

$$Y = A + \int_0^1 w(t)dt$$

It follows that $y$ is a Gaussian random variable with $E[Y|1] = A$, and variance

$$\sigma^2_{Y|1} = E[\int_0^1 \int_0^1 w(t)w(u)dtdu] = \int_0^1 \int_0^1 \frac{1}{2}\delta(t-u)dtdu = \frac{1}{2}$$

Therefore

$$f_Y(y|1) = \frac{1}{\sqrt{\pi}} \exp(-(y-A)^2) \tag{2}$$

Similarly, when a 0 is transmitted

$$Y = 0 + \int_0^1 w(t)dt,$$

and it follows that $y$ is a Gaussian random variable with $E[Y|0] = 0$, and variance

$$\sigma^2_{Y|0} = E[\int_0^1 \int_0^1 w(t)w(u)dtdu] = \int_0^1 \int_0^1 \frac{1}{2}\delta(t-u)dtdu = \frac{1}{2}$$
Therefore

\[ f_Y(y|0) = \frac{1}{\sqrt{\pi}} \exp(-y^2) \]

(3)

Using equations (2) and (3) in equation (1), we get

\[ 1 = \frac{f_Y(\lambda_{opt} | 1)}{f_Y(\lambda_{opt} | 0)} = \frac{\exp(-(\lambda_{opt} - A)^2)}{\exp(-\lambda^2_{opt})} \]

Taking log on both sides and rearranging, we get

\[ \lambda^2_{opt} = (\lambda_{opt} - A)^2 \]

\[ \Rightarrow \lambda_{opt} = A/2. \]

(b) Using the threshold in part (a), i.e., \( \lambda = A/2 \), we can evaluate the average probability of error for this receiver in terms of the the complementary error function \( \text{erfc}(x) \) as follows:

Consider a zero being transmitted, then the conditional probability of making an error is

\[ P_{e0} = P(y > A/2 | 0) = \int_{A/2}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-y^2) = \frac{1}{2} \text{erfc}(A/2) \]

By symmetry it follows that \( P_{e1} = P_{e0} \Rightarrow P_e = P_{e1} = P_{e0} = \frac{1}{2} \text{erfc}(A/2) \)

2. The \( Q \) function is defined as

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp(-z^2/2)dz \]

Now apply a change of the variable \( z \) as \( z = \sqrt{2}t \). Then it follows that

\[ Q(x) = \frac{1}{2} \text{erfc}(\frac{x}{\sqrt{2}}) \]

3. The analog signal is sampled at \( f_s = 8 \text{KHz} \). Each sample is represented by 6 bits. Therefore the number of bits that need to be transmitted every second is

\[ R = 6f_s = 48 \text{Kbits/second} \]

For binary PCM transmission, the minimum transmission bandwidth required \( W \) is given as \( W = \frac{1}{2T} \), where \( T \) is the bit duration. Therefore,

\[ T = \frac{1}{R} \Rightarrow W = R/2 = 48/2 \text{Khz} = 24 \text{Khz} \]

4. \( X \) is a random variable with PDF given as

\[ f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-2)^2}{8}} \quad -\infty \leq x \leq \infty \]

Clearly \( X \) is Gaussian with mean \( \mu_X = 2 \) and variance \( \sigma^2_X = 4 \). Since a linear transformation of the Gaussian random variable results in a Gaussian random variable, \( Y = 2X + 10 \) is also Gaussian with \( \mu_Y = 2\mu_X + 10 = 14 \) and \( \sigma^2_Y = 4\sigma^2_X = 16 \). The PDF of \( Y \) is

\[ f_Y(y) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{(y-14)^2}{32}} \quad -\infty \leq y \leq \infty \]