THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2006**

Quiz II

There are THREE questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **The Pulsing Passion of PAM:** Ok, now that I've got your attention, consider a pulse amplitude modulated system which uses a pulse shape $p(t) = \delta(t)$ so that the transmitted signal is

$$r(t) = \sum_k m(kT)\delta(t - kT)$$

where $m(t)$ is the program material.

(a) *(15 points)* Assume $m(t)$ has non-zero spectral energy on the whole interval $\pm W$ Hz and zero spectral energy elsewhere. Furthermore, assume the spectrum of $m(t)$, $M(f)$ is continuous. Assume some spectral shape for $M(f)$, the Fourier transform of $m(t)$, and carefully sketch $R(f)$, Fourier Transform of $r(t)$.

SOLUTION: *It's just a set of copies of $M(f)$ centered at integral multiples of $1/T$.*

(b) *(15 points)* What is the maximum allowable value of sampling interval T which allows $m(t)$ to be recovered uncorrupted from $r(t)$?

SOLUTION: *The spectra overlap SOMEWHERE unless $T \leq 1/2W$ – the Nyquist sampling period.*

(c) *(20 points)* Now, suppose that

$$M(f) = [u(f + W) - u(f + W/2)] + [u(f - W/2) - u(f - W)]$$

What is the maximum value of sampling interval T which will allow $m(t)$ to be recovered from $r(t)$? If we can recover $m(t)$ from $r(t)$, show exactly how it might be done.

SOLUTION: *If you sketch the spectra $M(f)$ you'll see that if $M(f)$ is shifted to the left or right by W it "nests" inside the original $M(f)$ and does not overlap. Therefore one can use a sampling interval of $T = 1/W$. To recover the signal, we apply a filter shaped exactly like $M(f)$ at the receiver.*

Note that making T slightly different from $1/W$ screws things up – the spectra overlap. So, we still have, in general, $T \leq 1/2W$. But owing to the particular shape of the $M(f)$ we can also get away with exactly $T = 1/W$. This method is called "subsampling" and the possible subsampling rates are intimately tied to the characteristics of the signal being sampled.

2. *(50 points)* **Quantization:**

   We have simply assumed that for an optimum quantizer, the $\{q_k\}$ are *ordered* – that is $q_k > q_{k-1}$. In this problem we will prove that the $\{q_k\}$ MUST be ordered from smallest to largest.

   Please note that the bin boundaries $\{x_k\}$ are by definition ordered since we can always number successive boundaries that way. Thus, our derivation of the Loyd-Max rules is still valid since no ordering was assumed on the q_k in the derivation. The Loyd-Max rules are

$$q_k = E\left[X | X \in (x_{k-1}, x_k)\right]$$

   and

$$x_k = \frac{q_{k+1} + q_k}{2}$$

   Please prove that $q_{k+1} > q_k$ for all quantizer levels in an optimum quantizer.

   SOLUTION: *By definition, the $\{x_k\}$ are ordered. Thus, for any $q \in (x_k, x_{k+1})$ and $q' \in (x_{k-1}, x_k)$ we must have $q > q'$. Since $q_k \in (x_{k-1}, x_k)$ and $q_{k+1} \in (x_k, x_{k+1})$ (the conditional expectation has to reside in the conditioned interval) we must have $q_{k+1} > q_k$. Q.E.D.*

3. *(50 points)* **Cora The Terminator:** Cora the communications engineer has been hired by SquirrelNOT, Inc. to predict the future positions of a particularly pesky squirrel named Martin P. Sciuridae. Given a sequence of his last known locations $\{s_n\}$, Cora is charged to develop a linear predictor for his position at the next time step. The company plans to use it's patented squirrel zapper at the predicted location. If Martin is there, he's history, but if not, such an attempt will just make him angry.

   And there is nothing worse than an angry squirrel.

   The form of Cora's linear predictor is

$$\hat{s}_n = \sum_{k=1}^{M} s_{n-k} w_k$$

   where the w_k are constants. You're going to help Cora find the appropriate $\{w_k\}$ to minimize a mean square error criterion

$$\text{MSE} = E\left[\left(s_n - \sum_{k=1}^{M} s_{n-k} w_k\right)^2\right]$$

   WARNING: This is NOT the same problem as in Quiz II 2004.

   (a) *(20 points)* Marty is a creature of habit and his position follows

$$s_{n+1} = \sum_{k=0}^{N-1} a_k s_{n-k}$$

   where the $\{a_k\}$ are constants. Find the weights $\{w_k\}$ which minimize the MSE. How many weights do you need? You must justify your result.

SOLUTION: *s_{n+1} is explicitly given in terms of the previous N values of s_n. So, $w_k = a_{k+1}$ and there are $M = N$ weights Sanity check:*

$$\hat{s}_n = \sum_{k=1}^{N} s_{n-k} a_{k+1} = \sum_{\ell=0}^{N-1} s_{(n-1)-\ell} a_\ell = s_n$$

(b) *(10 points)* Now suppose that Marty follows

$$s_{n+1} = -2s_n - s_{n-1} \tag{1}$$

What are the $\{w_k\}$ which minimize the MSE? As before you must justify your result.

SOLUTION: *By the previous part, $w_1 = -2$ and $w_2 = -1$.*

(c) *(20 points)* Now suppose Cora does not know the a_k initially and only can observe Marty's position s_n. Cora decides to use an adaptive procedure – calculating the error gradient and adjusting the weights in the opposite direction:

$$w_j(n+1) = w_j(n) + \mu e_n s_j$$

where

$$e_n = s_n - \sum_{k=1}^{M} s_{n-k} w_k(n)$$

and $\mu > 0$.

You may assume that Cora knows $M = 2$. Assume that $s_n = (-1)^n$ (verify that this satisfies the difference equation (1)). Please expand and then simplify the weight adaptation equation above to show that in this degenerate case, Cora will probably lose her job. Why do we say this case is degenerate? What if we assume $M = 1$?

SOLUTION:

$$w_j(n+1) = w_j(n) + \mu(-1)^j \left((-1)^n - \sum_{k=1}^{2} (-1)^{n-k} w_k(n) \right)$$

and factoring out $(-1)^n$ to get

$$w_j(n+1) = w_j(n) + \mu(-1)^{j+n} \left(1 - \sum_{k=1}^{2} (-1)^{-k} w_k(n) \right)$$

We can then separate out by weights

$$w_1(n+1) = w_1(n) - \mu(-1)^n \left(1 - \sum_{k=1}^{2} (-1)^{-k} w_k(n) \right)$$

and

$$w_2(n+1) = w_2(n) + \mu(-1)^n \left(1 - \sum_{k=1}^{2} (-1)^{-k} w_k(n) \right)$$

Now, fully expanding we have

$$w_1(n+1) = w_1(n) - \mu(-1)^n \left(1 + w_1(n) - w_2(n)\right)$$

and

$$w_2(n+1) = w_2(n) + \mu(-1)^n \left(1 + w_1(n) - w_2(n)\right)$$

First we note that by adding the two equations we get

$$w_1(n+1) + w_2(n+1) = w_1(n) + w_2(n)$$

so the sum of the weights remains constant throughout the iterations. Thus, unless Cora guesses $w_1 + w_2 = 3$ she's screwed. We then notice that the full homogeneous solution to the difference equation is $A(-1)^n + Bn(-1)^n$, so that damn squirrel is moving in only one of the eigenmodes of the system, the degenerate!!!! (That's a pun, by the way – when a system uses only one of its possible modes, that's a "degenerate" solution.)

Now, suppose we use $M = 1$. We then have

$$w_1(n+1) = w_1(n) - \mu(-1)^n \left(1 + w_1(n)\right)$$

or

$$w_1(n+1) = w_1(n)\left(1 - \mu(-1)^n\right) - \mu(-1)^n$$

and we notice that a solution to this equation is $w_1(n) = -1$. This works great since it implies $\hat{s}_{n+1} = -s_n$ and this is exactly the case here since $s_n = (-1)^n$.