THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2005**

Quiz II

There are THREE questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Pulse Amplitude Modulation:** Consider the signal

$$r(t) = m(t) \sum_{k=-\infty}^{\infty} p(t-kT)$$

where $m(t)$ is program material bandlimited to $\pm W$ Hz and $p(t)$ is an arbitrary waveform such that the sum $\sum_{k=-\infty}^{\infty} p(t-kT)$ exists.

(a) *(20 points)* Suppose $p(t) = \delta(t)$. What condition on T insures that $m(t)$ can always be recovered from $r(t)$?

SOLUTION: *This is the form used to derive the Nyquist sampling theorem. We must have* $T < 1/2W$.

(b) *(20 points)* Now suppose $W = 10$, $T = 10^{-3}$, and $p(t) = \frac{sin\pi t}{t}$. Since $p(t)$ exists for all time, you'll notice that the pulses $p(t)$ which comprise $\sum_{k=-\infty}^{\infty} p(t-kT)$ OVERLAP. If we apply an ideal band pass filter $H(f) = u(f+1010) - u(f+990)u(f-990) - u(f-1010)$ (where $u()$ is the unit step function) to $r(t)$, show how can $m(t)$ be recovered from $r(t)$ (or not).

SOLUTION: *From class we know that the pulse sum, since it's periodic, is going to be a set of impulses in frequency separated by* $1/T$ *and scaled by the spectrum of* $p(t)$. *This can be easily derived by noting* $\sum_{k=-\infty}^{\infty} p(t-kT) = p(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT)$ *We know that the Fourier transform of* $\sum_{k=-\infty}^{\infty} \delta(t-kT)$ *is* $\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f-k/T)$. *Convolution in time domain implies multiplication in frequency domain so* $\mathcal{F}\left[p(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT)\right] = P(f)\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f-k/T)$

However, the spectrum of $p(t)$ *(a sinc pulse) in this case is nonzero only for* $|f| \leq 1$. *So, the spectrum of* $r(t)$ *is a scaled replica of* $M(f)$. *Since* $m(t)$ *is band limited to* ± 10 *Hz, the output of the bandpass filter is identically zero and* $m(t)$ *cannot be recovered.*

(c) *(20 points)* Now suppose $W = 10$, $T = 10^{-3}$, and $p(t) = \frac{sin10^5\pi t}{t}$. Since $p(t)$ exists for all time, you'll notice that the pulses $p(t)$ which comprise $\sum_{k=-\infty}^{\infty} p(t-kT)$ OVERLAP. If we apply an ideal band pass filter $H(f) = u(f+1010) - u(f+990)u(f-990) - u(f-1010)$ (where $u()$ is the unit step function) to $r(t)$, show how can $m(t)$ be recovered from $r(t)$ (or not).

SOLUTION: *From class we know that the pulse sum, since it's periodic, is going to be a set of impulses in frequency separated by* $1/T$ *and scaled by the spectrum of* $p(t)$.

However, the spectrum of $p(t)$ (a sinc pulse) in this case is nonzero only for $|f| \leq 10^5$. So, the spectrum of $r(t)$ is a bunch of scaled shifted replicas of $M(f)$ each centered at $f = 10^3 k$ where k is an integer. Since $m(t)$ is band limited to ± 10 Hz, the output of the bandpass filter is proportional to $M(f+10^3)+M(f-10^3)$ and we can use an envelope detector (or synchronous AM demodulation to recover $m(t)$.

2. *(50 points)* **Quantization:**

   (a) *(20 points)* What is the purpose of a quantizer? State your answer in words (no more than a short paragraph). NOTE: this is not an *optimality* question, just a simple question about what a quantizer is used for.

   SOLUTION: *The purpose of a quantizer is to approximate samples (usually of a waveform) using a* finite *set of amplitude levels. Such quantization is a precursor for digital transmission of a signal since samples of a continuous real-valued waveform cannot otherwise be represented with a finite number of bits.*

   (b) *(30 points)* We have seen in class that an optimal quantizer function $Q(x)$ seems to always be non-decreasing. Please PROVE that this observation is true (or not). You may assume that the PDF of X, the random variable to be quantized, exists and is nonzero for all $X \in \Re$. HINT: start from the Loyd-Max conditions and remember how these conditions were derived.

   SOLUTION: *For quantization levels q_k and bin dividers x_k, we have Loyd-Max:*

$$x_k = \frac{q_{k+1}+q_k}{2}$$

   and

$$q_k = E\left[X | X \in [x_{k-1}, x_k)\right]$$

   We need to prove that $q_{k+1} \geq q_k$ or

$$E\left[X|X \in [x_k, x_{k+1})\right] - E\left[X|X \in [x_{k-1}, x_k)\right] \geq 0$$

   Well, by definition, the bin-dividers x_k are ordered from smallest to largest. That is $x_{k+1} \geq x_k$. Therefore, the intervals $[x_{k-1}, x_k)$ and $[x_k, x_{k+1})$ are consecutive and disjoint. Thus, for any PDF on X we must have

$$E\left[X|X \in [x_k, x_{k+1})\right] \in [x_k, x_{k+1})$$

   and

$$E\left[X|X \in [x_{k-1}, x_k)\right] \in [x_{k-1}, x_k)$$

   which immediately implies

$$E\left[X|X \in [x_k, x_{k+1})\right] \geq E\left[X|X \in [x_{k-1}, x_k)\right]$$

   and thus $q_{k+1} \geq q_k$.

3. *(50 points)* **Cora and the Quantizer/Coder From Hell:**

   Cora the communications engineer has been hired by Mephisto Incorporated to design a communications hot line (tee hee) for the Prince of Darkness himself. The Prince wishes to use the hot line to remotely measure the temperature, $x(t)$ in various parts of his domain. As one might imagine, the sample sequence for the temperature is constantly increasing. In fact in one particular area the sampled temperature follows

$$x_n = n/2$$

Assume Cora needs to encode and transmit this sequence.

Cora has a choice of two systems. The block diagram for the first scheme is given in FIGURE 1. Basically, a direct difference is computed for the input signal x_n and input to a 1-bit quantizer. A coder then outputs a binary 1 or 0 depending upon whether the quantizer output is $+1$ or -1 respectively. At the receiver, the 1's and 0's are converted into ± 1s and cumulatively summed to obtain $\hat{x}_n$.

The block diagram for the second system is shown in FIGURE 2.

In this problem we will evaluate the effectiveness of both systems. For all parts assume that $\hat{y}_0 = 1$ and $q_n = 0$ for $n < 0$.

(a) *(10 points)* For system A in FIGURE 1, sketch the discrete sequence $\hat{y}_n$ for $n = 0, 1 \ldots 10$. What is the corresponding binary code sequence?

SOLUTION: *The first coder only codes the difference directly, and the difference is always positive. Thus, the coder output is* $\hat{y}_n = 1$ *and the binary output is* $111111\ldots$.

(b) *(10 points)* For system B in FIGURE 2, write down expressions for y_n and q_n by analyzing the block diagram and then sketch the discrete sequence $\hat{y}_n$ for $n = 0, 1 \ldots 10$. What is the corresponding binary code sequence?

HINT: It might help to put everything in a table.

SOLUTION: *This coder does not blindly look at the difference. It takes into account the errors made by the coder by trying to predict* x_{n-1} *and transmit that difference; i.e., you have a negative feedback loop!*

The equations we need to turn the crank are:

$$y_n = x_n - q_{n-1}$$

where q_{n-1} *is the input to the first adder and*

$$q_n = \hat{y}_n + q_{n-1}$$

Assuming initial rest ($q_n = 0$ *for* $n < 0$*) we have:*

n	x_n	y_n	$\hat{y}_n$	q_n	b_n
0	0	0	1	1	1
1	1/2	-1/2	-1	0	0
2	1	1	1	1	1
3	3/2	1/2	1	2	1
4	2	0	1	3	1
5	5/2	-1/2	-1	2	0
6	3	1	1	3	1
7	7/2	1/2	1	4	1
8	4	0	1	5	1
9	9/2	-1/2	-1	4	0
10	5	1	1	5	1
11	11/2	1/2	1	6	1
12	6	0	1	7	1
13	13/2	-1/2	-1	6	0
14	7	1	1	7	1
15	15/2	1/2	1	8	1
16	8	0	1	9	1

(c) *(10 points)* For system A, carefully sketch the resulting $\hat{x}_n$. You may assume that $\hat{x}_0 = 0$.

SOLUTION: *Sketch here is simple. Only 1's are transmitted and that corresponds to always adding increments of* $+1$. *So* $\hat{x}_n = n$.

(d) *(10 points)* Repeat the previous part for system B. Comment on any differences you find between the outputs generated by the two methods. Which, if either, does a better job? Why?

SOLUTION: *In words, system B works out to, 3-up, 1-down, 3-up, 1-down for an average of 2-up every four steps; i.e., a slope of 1/2 just like we want.*

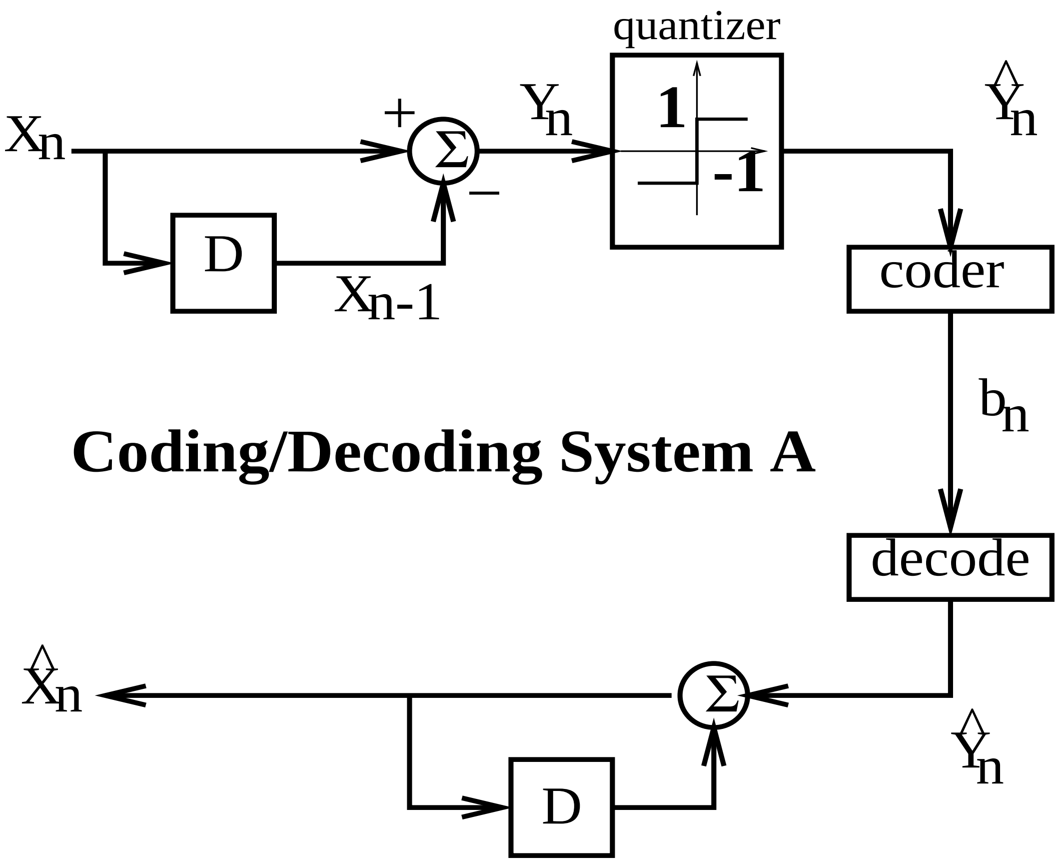

Figure 1: System A for problem 3

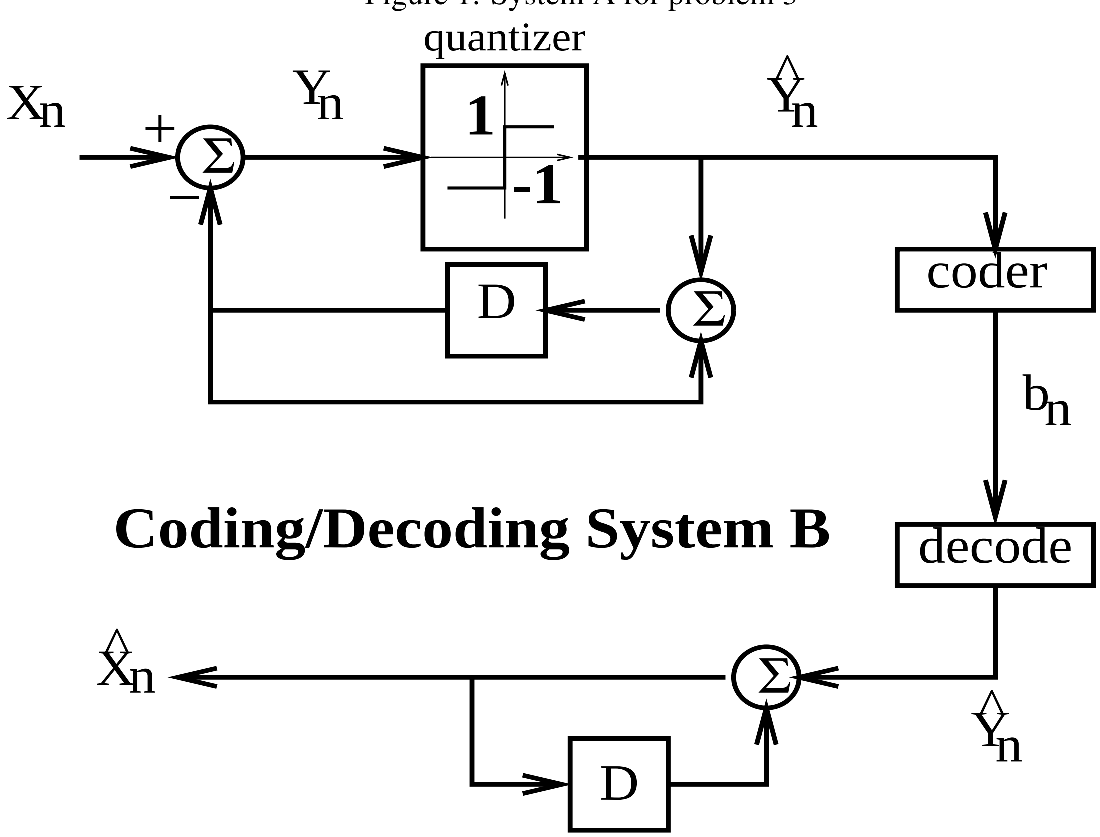

Figure 2: System B for problem 3