THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring**
Problem Set 6

Reading:

Review Probability (appendix and 322:321 notes/text)
Web notes on convexity

1. **Counting and Sample Space:** Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is zero with probability 0.8, independent of any other bit.

   (a) What is the probability of the code word 00111?

   SOLUTION: *Since the probability of a zero is* 0.8*, we can express the probability of the code word 00111 as two occurences of a 0 and three occurences of a 1. Therefore*

   $$P[00111] = (0.8)^2(0.2)^3 = 0.00512$$

   (b) What is the probability that a code word contains exactly three ones?

   SOLUTION: *The probability that a code word has excatly three 1's is*

   $$P[three\ 1's] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512$$

2. **Simple Probability, Simple Application:** A source wishes to transmit radio packets to a receiver over a radio link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is sent error free the receiver sends an acknowledgement(ACK) back to the source. When the receiver gets a packet with errors, it sends back a negative-acknowledgement(NACK). Each time the source receives a NACK, the packet is re-transmitted. We assume that each packet transmission is independently corrupted by errors with probability q.

   (a) Find the PMF of X, the number of times that a packet is transmitted by the source.

   SOLUTION: *The source continues to transmit packets until one is received correctly. Hence the total number of times a packet is transmitted is* $X = x$*, if the first* $x - 1$ *transmissions were in error. Therefore the PMF of* X *is*

   $$P_X(x) = \begin{cases} q^{(x-1)}(1-q) & x = 1, 2, .. \\ 0 & \textit{otherwise} \end{cases}$$

   (b) Suppose each packet takes 1 millisecond to transmit and that the source waits an additional millisecond to receive the acknowledgement message (ACK or NACK) before transmitting. Let T equal the time required until the packet is successfully received. What is the relationship between T and X? What is the PMF of T?

SOLUTION: *The time required to send a packet is 1 millisecond and the time required to send an acknowledgement back takes another millisecond. Thus if X transmissions of a packet are required to send a packet correctly, then the packet is correctly received after $T = 2X - 1$ milliseconds. Therefore, for an odd integer $t > 0$, $T = t$ iff $X = \frac{t+1}{2}$. Thus,*

$$P_T(t) = P_X(\frac{t+1}{2}) = \begin{cases} q^{\frac{t-1}{2}}(1-q) & t = 1, 3, 5, .. \\ 0 & \textit{otherwise} \end{cases}$$

3. **Joint PMFs:** Calls arriving ar a telephone switch are either voice calls (v) or data calls (d). Each call is a voice call with probability p, independent of any other call. Observe calls at a telephone switch until you observe two voice calls. Let M equal the number of calls up to and including the first voice call. Let N equal the number of calls observed up to and including the second voice call. Find the conditional PMF's $P_{M|N}(m|n)$ and $P_{N|M}(n|m)$. Interpret your results.

   SOLUTION: *The key to solving this problem is to find the joint PMF of M and N. Note that $N \geq M$. For $n \geq m$, the joint event $\{M = m, N = n\}$ has probability,*

$$\begin{aligned} P[M = m, N = n] &= (1-p)^{(m-1)}p(1-p)^{(n-m-1)}p \\ &= (1-p)^{(n-2)}p^2 \end{aligned}$$

   A complete expression for the joint PMF of M and N is

$$P_{M,N}(m,n) = \begin{cases} (1-p)^{(n-2)}p^2 & m = 1, 2, .., n-1; n = m+1, m+2, .. \\ 0 & \textit{otherwise} \end{cases}$$

   For $n = 2, 3, ..$, the marginal PMF of N satisfies,

$$P_N(n) = \sum_{m=1}^{n-1}(1-p)^{(n-2)}p^2 = (n-1)(1-p)^{(n-2)}p^2$$

   Similarly, for $m = 1, 2, ..$, the marginal PMF of M satisfies,

$$\begin{aligned} P_M(m) &= \textstyle\sum_{n=m+1}^{\infty}(1-p)^{(n-2)}p^2 \\ &= p^2[(1-p)^{(m-1)} + (1-p)^m + ..] \\ &= (1-p)^{(m-1)}p \end{aligned}$$

   Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{(n-m-1)}p & n = m+1, m+2, .. \\ 0 & \textit{otherwise} \end{cases}$$

   The interpretation of the conditional PMF of N given M is that given $M = m$, $N = m + N'$, where N' has geometric PMF with mean $1/p$. The conditional PMF of M given N is,

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} \dfrac{1}{n-1} & m = 1, 2, .., n-1 \\ 0 & \textit{otherwise} \end{cases}$$

Given that call $N = n$ was the second voice call, the first voice call is equally likely to occur in any of the previous $n-1$ calls.

4. **Gaussians:** W is a Gaussian random variable with expected value $\mu = 0$ and variance $\sigma^2 = 16$. Given the event $C = \{W > 0\}$,

   (a) What is the conditional PDF $f_{W|C}(w)$?
   SOLUTION:
   $$f_W(w) = \frac{1}{\sqrt{32\pi}} \exp\left(-w^2/32\right)$$
   Since W has expected value $\mu = 0$, $f_W(w)$ is symmetric about $w = 0$. Hence $P[C] = P[W > 0] = 0.5$.
   $$f_{W|C}(w) = \begin{cases} \dfrac{f_W(w)}{P[C]} & w \in C \\ 0 & \textit{otherwise} \end{cases} = \begin{cases} \dfrac{2}{\sqrt{32\pi}} exp(-w^2/32) & w > 0 \\ 0 & \textit{otherwise} \end{cases}$$

   (b) What is the conditional expected value E$[W|C]$?
   SOLUTION: *The conditional expected value of W given C is*
   $$E[W|C] = \int_{-\infty}^{\infty} w f_{W|C}(w)\, dw = \frac{2}{4\sqrt{2\pi}} \int_0^{\infty} w exp(-w^2/32)\, dw$$
   Making substitution $v = w^2/32$, we obtain
   $$E[W|C] = \frac{32}{\sqrt{32\pi}} \int_0^{\infty} exp(-v)\, dv = \frac{32}{\sqrt{32\pi}}$$

   (c) Find the conditional variance, Var$[W|C]$.
   SOLUTION:
   $$E[W^2|C] = \int_{-\infty}^{\infty} w^2 f_{W|C}(w)\, dw = 2\int_0^{\infty} w^2 f_W(w)\, dw$$
   We observe that $w^2 f_W(w)$ is an even function. Hence
   $$E[W^2|C] = 2\int_0^{\infty} w^2 f_W(w)\, dw = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = E[W^2] = 16$$
   Lastly conditional variance of W given C is
   $$Var[W|C] = E[W^2|C] - (E[W|C])^2 = 16 - \frac{32}{\pi} = 5.81$$

5. **Functions of random variables:**

   (a) Find the PDF $f_Y(y)$ for the random variable Y, where $Y = X^3$ and X is a uniform random variable in the range (0,1).
   SOLUTION: *This transformation is a one-to-one mapping where if $Y = y$ then corresponding $X = y^{1/3}$. CDF of random variable Y can be written as,*
   $$F_Y(y) = Prob[Y \le y] = Prob[X \le y^{(1/3)}] = F_X(y^{1/3}) = \int_{x=0}^{y^{1/3}} 1\, dx = y^{1/3}$$
   $$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{1}{3} y^{-2/3}$$

(b) Let X and Y be independent random variables with $f_X(x) = \exp(-x)u(x)$, and $f_Y(y) = \frac{1}{2}[u(y+1) - u(y-1)]$ and let $Z = X + Y$, where $u()$ denotes unit step function. What is the PDF of random variable Z? HINT:Use convolution!

SOLUTION: *Since Z is the sum of 2 independent random variables, the PDF of Z can be obtained by convolving PDF's of X and Y.*

$$f_Z(z) = \int f_X(z-y) f_Y(y)\, dy$$

$f_X(z-y) = \exp -(z-y)u(z-y)$ *For different regions of z values we get different integrals as follows*

$z < -1$,

$$f_Z(z) = 0$$

$-1 \leq z < 1$

$$f_Z(z) = \frac{1}{2}\int_{-1}^{z} \exp\left[-(z-y)\right] dy = \frac{1}{2}[1 - \exp\left[-(z+1)\right]]$$

$z \geq 1$

$$f_Z(z) = \frac{1}{2}\int_{-1}^{1} \exp\left[-(z-y)\right] dy = \frac{1}{2}[\exp\left[-(z-1)\right] - \exp\left[-(z+1)\right]]$$