THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring**

Problem Set 2

1. **When Is a Carrier a Good Carrier?:** Given an information signal $m(t)$ you want to generate

$$r(t) = Am(t)\cos(2\pi f_c t) \tag{1}$$

However, life (your professor) is unkind and you're only allowed to use $\cos^3(2\pi f_c t)$ as your carrier signal. That is, the first stage of your transmitter block diagram with $m(t)$ as the input is going to be a multiplication by $\cos^3(2\pi f_c t)$ (instead of the usual sane $\cos(2\pi f_c t)$).

(a) Assume you can build any linear time invariant (LTI) filter you'd like. Can you filter the output of the multiplier to obtain the desired signal? If so, what is the filter characteristic?

SOLUTION: *In order to obtain $r(t) = A_c m(t)\cos(2\pi f_c t)$,we need to design a filter that give us the desired signal, therfore using the math identities such as:*

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$$

$$\cos^3(x) = \cos(x)\cos^2(x) = \cos(x)(\frac{1}{2} + \frac{1}{2}\cos(2x)) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$$

we can design a LPF filter so that we get rid off of the third harmonic components e.g. $\frac{1}{4}\cos(3(2\pi f_c))$ thus that we are only left with $\frac{3}{4}\cos(2\pi f_c t)$.

(b) Suppose $\cos^2(2\pi f_c t)$ is the carrier. Repeat the previous part.

SOLUTION: $\cos^2(2\pi f_c t)$ *does not have a component at frequency f_c. So, nope!*

(c) Suppose we generalize the carrier to be $\cos^n(2\pi f_c t)$ for $n > 2$. When can you generate the desired $r(t)$ using LTI filters.

SOLUTION: *Use the odd harmonics of the signal and we're always guaranteed a component at frequency f_c.*

2. **Simple Envelope Detection:** Consider the following AM signal

$$s(t) = A_c[1 + \lambda\cos(2\pi f_m t)]cos(2\pi f_c t) \tag{2}$$

The modulation factor is $\lambda = 1$ and $f_c \gg f_m$. The AM signal is applied to an ideal envelope detector producing the output $v(t)$.

(a) Why did we stipulate $f_c \gg f_m$?

SOLUTION: *An envelope detector works by allowing the incoming signal to quickly drag the envelope detector output higher, but then resisting negative swings via the diode (the output capacitor has to discharge). This means that an envelope detector tracks the peaks of the incoming signal and when the carrier frequency is high, the rectified signal looks like a "picket fence" representation of the low frequency information signal. If the carrier frequency is on the order of the frequencies in the information signal, there are no "peaks" to detect.*

(b) Find $v(t)$ and $V(f)$.

SOLUTION: *Just for reference we compute*

$$S(f) = \frac{1}{4}A_c \left\{ \begin{array}{c} 2\left[\delta(f+f_c)+\delta(f-f_c)\right] + \lambda\left[\delta(f+f_c+f_m)+\delta(f+f_c-f_m)\right] \\ + \\ \lambda\left[\delta(f-f_c+f_m)+\delta(f-f_c-f_m)\right] \end{array} \right\}$$

Now to the main show. A simple envelope detector chops off the negative half-cycles of the carrier. For cosine modulation and the information signal not going negative, this is equivalent to multiplying the signal $s(t)$ by an even square wave with the same period as the carrier. If we call this signal $z(t)$ we have analytically

$$z(t) = \sum_{k=-\infty}^{\infty} p(t - \frac{k}{f_c})$$

where

$$p(t) = \left[u(t + \frac{1}{4f_c}) - u(t - \frac{1}{4f_c})\right]$$

So, we then have

$$v(t) = s(t)z(t)$$

and

$$V(f) = (S * Z)(f)$$

Since $z(t)$ is periodic, we calculate its FT by first representing it as a Fourier Series

$$z_n = f_c \int_{-\frac{1}{4f_c}}^{\frac{1}{4f_c}} e^{-j2\pi f_c n t} dt = \frac{\sin\frac{\pi}{2}n}{\pi n}$$

so that

$$Z(f) = \sum_{n=-\infty}^{\infty} z_n \delta(f - nf_c)$$

We note that since $z(t)$ is even, the coefficients z_n are real.

Doing the convolution graphically (or in our heads since it's just impulses) we get copies of $S(f)$ at multiples of f_c so

$$V(f) = \sum_{n=-\infty}^{\infty} z_n S(f - nf_c)$$

which leaves us with a the baseband signal

$$\frac{1}{2}A_c z_1 \left\{2\delta(f) + \lambda\left[\delta(f+f_m) + \delta(f-f_m)\right]\right\}$$

and a bunch of upper sidebands. The LPF filters out these upper sidebands and we're left with only the baseband signal which is simply our sinusoid with a DC offset.

(c) How does your answer change (qualitatively) if $\lambda > 1$?

SOLUTION: *The program material now goes negative and the rectifier makes it positive. There's no difference in $S(f)$, but now we can't just multiply by $z(t)$ to get the post-rectified signal.*

3. **More Modulation Hijinks:** A switching modulator (refer to figure P2.3 page 167 in your text) uses a carrier wave $c(t)$ to generate a modulated signal. The diode acts like an ideal switch described by:

$$v_2(t) = \begin{cases} v_1(t) & \text{if } c(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine $v_2(t)$ and $V_2(f)$

SOLUTION: *$v_2(t)$ represents the change in voltages for the carrier signal. i.e if the carrier amplitude voltage is greater than the modulating signal voltage $m(t)$, $v_1(t)$ is a nonzero voltage and thefore the diode conducts. However, if the carrier voltage is less than the modulating signal $m(t)$, the diode switches off and $v_2(t) = 0$. We first find*

$$v_1(t) = A_c\cos(2\pi f_c t) + m(t)$$

simply by kirchoff's voltage law. At this point you're saying "Hey! That's not a modulated wave!!!! You've just offset the sinusoid by $m(t)$!" but wait just a second.

Once we rectify, we have what looks like the output (before filtering) of a rectifier for an envelope detector. That is, we lopped off the negative portion of $v_1(t)$ which made it not look like a real (symmetric) envelope.

Approximately (assuming the drop across the diode is small and $m(t)$ varies SLOWLY compared to $\cos 2\pi f_c t$ we have

$$v_2(t) \simeq (A_c\cos(2\pi f_c t) + m(t))g_{T_o}(t)$$

where

$$g_{T_0}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}cos(2\pi f_c t(2n-1))$$

Though this looks strange, it's just the formal way of writing down the positive caps of the cosine carrier in a Fourier series. This will be useful later when we want to identify components at the carrier frequency. You should verify that it's correct by calculating the fourier series of $[\cos 2\pi f_c t]^+$ where the $[x]^+$ notation means zero unless $x \geq 0$.

The fourier transform of $v_2(t)$ is

$$V_2(f) = \frac{A_c}{2}(\delta(f-f_c) + \delta(f+f_c)) * G_{T_o}(f) + M(f) * G_{T_o}(f)$$

Note: $(*)$ represents convolution

(b) Find the AM wave component in $v_2(t)$ as follows

SOLUTION: *What you're looking for is something of the form* $(1+k_a m(t))\cos 2\pi f_c t$. *So now you have to look more carefully at the fourier series representation.*

$$v_2(t) = (A_c \cos(2\pi f_c t) + m(t)) \left(\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} cos(2\pi f_c t(2n-1)) \right)$$

The second term in the sum is where the action is – the carrier is the $n = 1$ *term. Terms for* $n > 1$ *have frequencies greater than* f_c. *And the constant* $m(t)$ *term is low frequency so we can throw it away (filter it out) just like we filter out the higher frequencies where* $n > 1$. *So we can write*

$$s(t) = A_c \cos(2\pi f_c t) + m(t)\frac{2}{\pi}\cos 2\pi f_c t$$

Regrouping we have

$$s(t) = \left(1 + \frac{2}{\pi A_c} m(t)\right) \cos(2\pi f_c t)$$

which is the standard form for an AM signal.

(c) Identify the unwanted components in $v_2(t)$ at the output of the demodulator

SOLUTION: *We essentially did this above. You just filter them out.*

(d) A modulation signal $m(t)$ and sent it through a vestigial sideband modulator that outputs $s(t)$. Find $S(f)$. Design a proper demodulator for this modulated output.

SOLUTION: *There are several ways to do this. I prefer to do it simply given the wording of the problem. 1) first you modulate the signal in the regular way (multiply by cosine) 2) Then you apply a filter with that special symmetric edge property. Call it* $H(f)$. *You're done.*

Now, you can also get fancier and do something akin to the Hilbert transform and modulate $m(t)$ *in cosine and an associated* $\tilde{m}(t)$ *in sine and add appropriately. But frankly, if you're going to go to all that trouble, you might as well just USE the Hilbert transform in the first place!*

Nonetheless, a more carefully worded version of this problem might not be a bad quiz I problem.

4. **Bandwidth Efficiency:** A normalized transmission bandwidth is defined by

$$\eta = \frac{B_T}{W} \tag{3}$$

where B_T is the transmission bandwidth of the modulated signal, and W is the message bandwidth. Compute the values of η for modulation schemes of AM, DSB-SC and SSB.

SOLUTION: *The bandwidth used by suppressed carrier AM (and large carrier AM for that matter) is* $2W$ *where* W *is the double sided bandwidth of the message signal. For single sideband, you use half that. For this definition of efficiency (which is pretty hokey – upside down, really) you then have efficiencies of* $\eta = 2$ *(SC and large carrier) and* $\eta = 1$ *for SSB.*

NOW, one can also talk about the POWER efficiency, but that's a signal to noise ratio issue which well deal with in the week before the exam (next week that is! WOW how time flies!).

5. **Cora and Carrier Squirrel:**

Cora the famous Rutgers Communications Engineer has been hired by Arboretums 'R Us to build an AM communications system for their small forest. Being somewhat eccentric, they have designed a signature component, *the squirrel carrier signal generator*, which consists of a tiny fast squirrel sitting on a heated switch. When the switch gets hot, the squirrel jumps which breaks the switch connection and shuts off the heater. When the squirrel lands, it turns the heater back on and the process begins again. If a given squirrel's evil twin is put in an identical box, it out of sheer meaness gets out of phase (it jumps up just before the other squirrel lands on its switch and comes down just before the other squirrel jumps up).

A complete up down cycle for each squirrel lasts $T = 1/f_c$ seconds. Since the switch is either closed or open, the carrier generator output is essentially binary and takes on values ± 1. The rest of the system components are more standard (electronic multipliers, LTI filters, etc.).

For mathematical convenience, we will define $c(x) = \text{sgn}\,(\cos(x))$ and $s(x) = \text{sgn}\,(\sin(x))$ with $\text{sgn}\,(0) = 1$.

(a) Carefully sketch $c(2\pi f_c t)$ and $s(2\pi f_c t)$ as a function of t for $f_c = 10$.

SOLUTION: $c()$ *just a symmetric zero mean square wave, centered about the origin with frequency* 10*Hz.* $s()$ *is offset to the right by a half cycle so it's an odd function.*

(b) Suppose Cora has program material $m(t)$ and uses it to modulate $c(2\pi f_c t)$ where $f_c = 10$. Carefully sketch the signal $r(t) = m(t)c(2\pi f_c t)$ for some $m(t)$ of your choosing which varies slowly as compared to $c(2\pi f_c t)$.

SOLUTION: *You've got a sqaure wave skeleton whose extrema outline* $m(t)$.

(c) Now suppose we form $r(t) = m(t)c(2\pi f_c t)$ Assume $c(2\pi f_c t)$ is available at the receiver (a synchronous squirrel system!). Show EXACTLY how $m(t)$ can be recovered at the receiver. Or if it cannot, show why not. As compared to modulation/demodulation using sinusoids, explain why a low pass filter is necessary (or unnecessary) to recover $m(t)$.

SOLUTION: *We note that* $c(2\pi f_c t)c(2\pi f_c t) = 1$. *So* $r(t)c(2\pi f_c t) = m(t)$ *and unlike regular AM, you get your original signal back simply by multiplying with the carrier. No LPF is necessary.*

(d) Of course, any good arboretum needs at least two independent channels – one for the trees and the other for the human visitors – but the squirrel modulator only comes in one frequency. So, suppose Cora gets normal – evil twin squirrel pairs and forms $r(t) = m_1(t)c(2\pi f_c t) + m_2(t)s(2\pi f_c t)$ where both $m_1(t)$ and $m_2(t)$ vary slowly as compared to $c(2\pi f_c t)$ and $s(2\pi f_c t)$. Assuming both $c(2\pi f_c t)$ and $s(2\pi f_c t)$ are available at the receiver (evil synchronous squirrels too!), show EXACTLY how $m_1(t)$ and $m_2(t)$ can be recovered. Or if they cannot, show why not.

Explain why a low pass filter is necessary (or unnecessary) to recover the $m_i(t)$.

SOLUTION: *You form two rails and obtain* $m_1(t) + m_2(t)s(2\pi f_c t)c(2\pi f_c t)$ *after multiplication by* $c(2\pi f_c t)$ *and* $m_2(t) + m_1(t)s(2\pi f_c t)c(2\pi f_c t)$ *after multiplication by* $s(2\pi f_c t)$. *In each case we have a baseband signal* $m_i(t)$ *and then an additive term which has only high frequency components – note that* $s()c()$ *is a square wave with DOUBLE the frequency of either* $s()$ *or* $c()$. *So by low pass filtering, we obtain the respective* $m_i(t)$.

(e) Suppose $r(t) = (1 + m(t))c(2\pi f_c t)$ where $|m(t)| \leq 1$. To recover $m(t)$ Cora full wave rectifies $r(t)$ to obtain $z(t) = |r(t)|$ and then applies an operator $T[]$ to $z(t)$. What operator $T[]$ should Cora use to recover $m(t)$? Is this operator linear?

SOLUTION: *We have* $1+m(t) \geq 0$ *and* $|c(x)| = 1$ *so* $|(1+m(t))c(2\pi f_c t)| = 1+m(t)$. *To recover* $m(t)$ *we just subtract* 1. *This operator is nonlinear since* $T[x] = x - 1$ *but* $T[2x] = 2x - 1$ *(and not* $2x - 2$*).*