THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2003**

Problem Set 1

1. Derive the convolution intergal from first principles (as outlined in class) given by $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$, where $h(t)$ is the impulse response of a LTI system, $x(t)$ the input and $y(t)$ the output.

   Convolution Integral: In time domain a linear system is described in terms of its impulse response which is defined as the response of the system (with zero initial conditions) to a unit impulse or delta funtion $\delta(t)$ applied to the input of a system. If the system is time invariant,then the shape of the impulse response is same no matter when the impulse is applied to the system.

   Let h(t) denote the impluse response of a LTI system. Let this system be subjected to an arbitrary excitation x(t). To determine the output y(t) we first approximate the input x(t) by staircase function composed of narrow rectangular pulses,each of duration $\Delta\tau$. The approximation becomes better for smaller $\Delta\tau$. As $\Delta\tau$ approaches zero, each pulse in the limit approaches a delta funtion weighed by a factor equal to the height of the pulse times $\Delta\tau$.

   Consider a pulse which occurs at $t = n\Delta\tau$. By defenition,the response of the system to a unit impulse or a delta funtion $\delta(t)$ occuring at $t = 0$ is h(t). Due to the time invariance property the response of the system to a delta funtion weighed by a factor $x(n\Delta\tau)\Delta\tau$ occuring at $t = n\Delta\tau$,must be

   $$\Delta y(t) = \lim_{\Delta\tau \to 0} x(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau$$

   Above is the response to one component of input occuring at $t = n\Delta\tau$. Since the system is linear we can apply superposition principle to find the response to the sum of the input components (which together constitute x(t)) occuring at different times as,

   $$y(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{n=\infty} x(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau$$

   The right hand side of the above equation by definition is the convolution integral that is,

   $$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

2. For each of the systems described by the input output relationships below, determine which of the following properties apply to the system: Memoryless(M), causal(C), linear(L), time-invariant(TI), stable(S). Justify your answers.

   (a) $y(t) = \sin(t+1)x(t)$

Memoryless: y(t) depends on the input only at time t,so the system is memoryless.

Time Invariant: We first compute output of the system and delay it by T,

$$z(t) = y(t-T) = sin(t-T+1)x(t-T)$$

Now delay bt T,and compute the output of the system

$$w(t) = sin(t+1)x(t-T)$$

Since $z(t) \neq w(t)$*,the system is time varying.*

Linear:First check Homogenity, the response to $ax(t)$*,*

$$sin(t+1)ax(t) = asin(t+1)x(t) = ay(t)$$

Next check additivity. Let $y_1(t)$ *be the response to* $x_1(t)$*and* $y_2(t)$ *be the response to* $x_2(t)$*. Then the response to* $x_1(t)+x_2(t)$ *is*

$$sin(t+1)[x_1(t)+x_2(t)] = sin(t+1)x_1(t)+sin(t+1)x_2(t) = y_1(t)+y_2(t)$$

Thus the system is linear.

Causal:The system is memoryless,so it is also causal.

Stable:Let the input x(t) be bounded for all t: $|x(t)| < K$ *for all t. Then,*

$$|y(t)| = |sin(t+1)x(t)| \leq |sin(t+1)||x(t)| \leq |x(t)| < K$$

So for bounded inputs,the output is also bounded and the system is stable.

(b) $y[n] = x[2-n]+1$

Memoryless: y[n] depends on the input at times other than n, so the system has memory.

Time Invariant:First compute output of the system and then delay it by N*,*

$$z[n] = y[n-N] = x[2-(n-N)]+1$$

We now delay it by N*, then compute output of the system.*

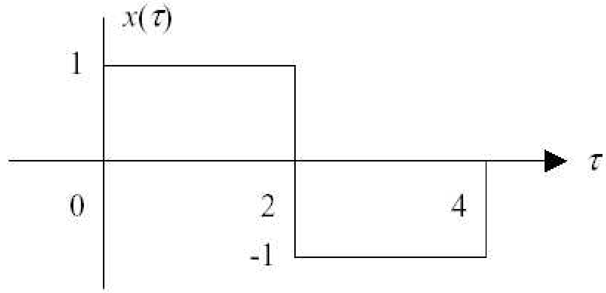
$$w[n] = x[2-n-N]+1$$

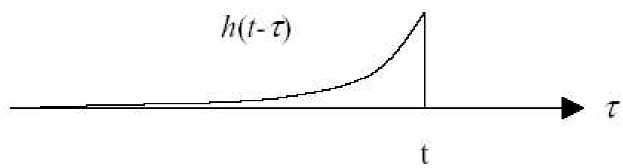
Since $w[n] \neq z[n]$*, the system is time varying.*

Linear:Let $y[n]$ *be the response to* $x[n] : y[n] = x[2-n]+1$*. We first check the homogenity-the response to* $ax[n]$*,*

$$a.x[2-n]+1 \neq a.y[n]$$

Homogenity is not satisfied and hence the system is not linear.





Causal:To get the output at $n = 0$, the system has to look at the input at time 2. Thus,the system is not causal.

Stable:Let the input $x[n]$ be bounded for all n: $|x[n]| < K$ for all n. Then,

$$|y[n]| = |x[2-n]+1| \leq |x[2-n]| + 1 < K + 1$$

So a bounded input implies bounded output,so the system is stable.

3. Consider a continuous time system with the following input $x(t)$ and impulse response $h(t)$,

$$x(t) = \begin{cases} 1 & 0 < t < 2 \\ -1 & 2 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

and $h(t) = \exp(-2t)u(t)$

(a) Compute the output of the system $y(t) = x(t) * h(t)$
Flipping and shifting $h(t)$ gives FIGURE 3a. From FIGURE 3a, we find the following different expressions depending on the value of t: $t < 0$. The non-zero portions of $x(\tau)$ and $h(t-\tau)$ do not overlap,so $y(t) = 0$. $0 \leq t \leq 2$,

$$\begin{aligned} y(t) &= \int_0^t 1.\exp(-2(t-\tau)\, d\tau \\ &= \exp(-2t) \int_0^t \exp(2\tau)\, d\tau \\ &= \tfrac{1}{2}(1-\exp(-2t)) \end{aligned}$$

$2 \leq t \leq 4$,

$$\begin{aligned} y(t) &= \int_0^2 1 \exp(-2(t-\tau))\, d\tau + \int_2^t -1 \exp(-2(t-\tau))\, d\tau \\ &= \tfrac{1}{2}([2\exp(4)-1]\exp(-2t)-1) \end{aligned}$$

$t \geq 4$,

$$\begin{aligned} y(t) &= \int_0^2 1 \exp(-2(t-\tau))\, d\tau + \int_2^4 -1 \exp(-2(t-\tau))\, d\tau \\ &= \tfrac{1}{2}\exp(-2t)(\exp(4)-1-\exp(8)+\exp(4)) \end{aligned}$$

(b) Is the system stable? Is the system causal?

The system is causal, since $h(t) = 0$ for $t < 0$. The system is stable, since $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^{\infty} exp(-2t)\, dt = \frac{1}{2} < \infty$

4. A signal $x(t)$ is periodic with period $T = 10^{-3}$. The Fourier series coeffecients for x(t) are given by

$$a_k = \begin{cases} (\frac{1}{2j})^k & k > 0 \\ 0 & k = 0 \\ (\frac{1}{-2j})^{-k} & \text{otherwise} \end{cases}$$

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Find the average power in the signal x(t), $\frac{1}{T}\int_0^T |x(t)|^2$. Hint:Use Parseval's relationship!

We use the Parseval's relationship:

$$\frac{1}{T}\int_0^T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

When $k > 0$

$$|a_k|^2 = |(\frac{1}{2j})^k|^2 = (\frac{1}{4})^k$$

When $k < 0$,

$$|a_k|^2 = |(\frac{1}{-2j})^{-k}|^2 = (\frac{1}{4})^{-k}$$

We can now find the average power:

$$\begin{aligned} \Sigma_{k=-\infty}^{\infty} |a_k|^2 &= \Sigma_{k=-\infty}^{k=-1} (\tfrac{1}{4})^{-k} + \Sigma_{k=1}^{k=\infty} (\tfrac{1}{4})^k \\ &= \Sigma_{i=1}^{i=\infty} (\tfrac{1}{4})^i + \Sigma_{k=1}^{k=\infty} (\tfrac{1}{4})^k \\ &= 2\Sigma_{k=1}^{k=\infty} (\tfrac{1}{4})^k \\ &= \tfrac{2}{3} \end{aligned}$$

5. Let $x(t) = \exp(-at)u(t)$,where u(t) is the unit step function. Find the Fourier transform of the following signals.

(a) $x(t)$

$$X(j\omega) = \int_0^{\infty} \exp(-at)\exp(-j\omega t)\, dt = \int_0^{\infty} \exp(-(a+j\omega)t\, dt) = \frac{1}{a+j\omega}$$

(b) $y(t) = x(t+5)$

Using the time shifting property of Fourier transform,

$$Y(j\omega) = e^{j\omega 5}.X(j\omega) = e^{j\omega 5}\frac{1}{(a+j\omega)}$$

(c) $z(t) = x(t)\sin(2\pi 40t)$

Writing z(t) as complex exponentials gives

$$z(t) = x(t)\frac{1}{2j}(\exp(j2\pi 40t) - \exp(-j2\pi 40t)) = \frac{1}{2j}x(t)\exp(j2\pi 40t) - \frac{1}{2j}x(t)\exp(-j2\pi 40t)$$

Applying frequency shitfing property of Fourier transform to each term of z(t) gives,

$$Z(j\omega) = \frac{1}{2j}[X(j(\omega - 2\pi 40t)) - X(j(\omega + 2\pi 40t))] = \frac{1}{2j}[\frac{1}{a + j(\omega - 2\pi 40)} - \frac{1}{a + j(\omega + 2\pi 40)}]$$

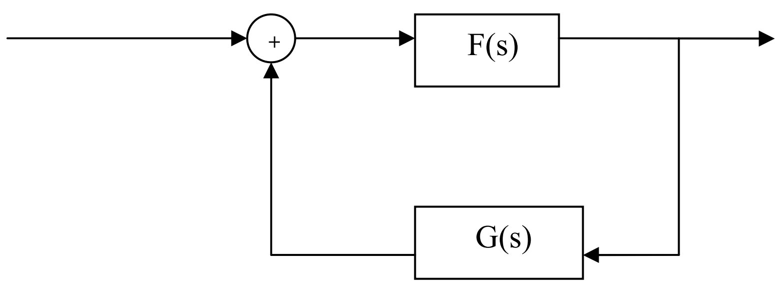
6. Evaluate the Fourier transform of the damped sinusoidal wave $g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$, where $u(t)$ is the unit step function.

$$\sin(2\pi f_c t) = \frac{1}{2j}[e^{j2\pi f_c t} - e^{-j2\pi f_c t}]$$

Therefore,applying the frequency shifting property to the Fourier transform pair we find that the Fourier transform of the damped sinusiodal wave is

$$G(f) = \frac{1}{2j}[\frac{1}{1 + j2\pi(f - f_c)} - \frac{1}{1 + j2\pi(f + f_c)}] = \frac{2\pi f_c}{(1 + j2\pi f)^2 + (2\pi f_c)^2}$$

7. Show that the overall system function $H(s)$ for the feedback system in FIGURE 7 is given by $H(s) = \frac{F(s)}{1 - F(s)G(s)}$.



$$W(s) = X(s) + G(s).Y(s), Y(s) = F(s).W(s) = F(s)[X(s) + G(s).Y(s)]$$

Re-arranging the previous equation,

$$F(s).X(s) = Y(s)[1 - F(s).G(s)]$$

System funtion

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(s)}{1 - F(s).G(s)}$$

8. A signal $x(t)$ of finite energy is applied to a sqaure-law device whose output is defined by $y(t) = x^2(t)$. The spectrum of $x(t)$ is limited to the frequency interval $-W \leq f \leq W$. Hence show that the spectrum of $y(t)$ is limited to $-2W \leq f \leq 2W$.

$$y(t) = x^2(t) = x(t)x(t)$$

Since multiplication in time domain corresponds to convolution in frequency domain, we may express the Fourier transform of y(t) as

$$Y(f) = \int_{-\infty}^{\infty} X(\lambda)X(f-\lambda)\,d\lambda$$

where X(f) is the Fourier transform of x(t). However X(f) is zero for $|f| > W$. *Hence,*

$$Y(f) = \int_{-W}^{W} X(\lambda)X(f-\lambda)\,d\lambda$$

In this integral we note that $X(f-\lambda)$ *is limited to* $-W \leq f-\lambda \leq W$. *When* $\lambda = -W$*,we find that* $-2W \leq f \leq 0$. *When* $\lambda = W$, $0 \leq f \leq 2W$. *Accordingly the Fourier transform Y(f) is limited to the frequency interval* $-2W \leq f \leq 2W$.