THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring**

Problem Set 11

Haykin: 5.1-5.7

1. **Cora and Squirrely Signal Design:** Cora the communications engineer has been given a set of signals:

$$s_1(t) = \cos\omega_c t + \cos 2\omega_c t$$
$$s_2(t) = \cos\omega_c t$$
$$s_3(t) = \cos\omega_c t - \cos 2\omega_c t$$
$$s_4(t) = \cos 2\omega_c t$$

for use on an interval $(0, T_b)$ where $T_b = 2\pi/\omega_c$. She needs to build a communication system around these signals to transmit her four (4) equally likely messages. However, her boss, Marty the Squirrel (how things change!) want's to know how well they'll work. Specifically, he wants to know the maximum probability of error when using a correlator receiver.

(a) Find a set of orthonormal functions to represent this set.

SOLUTION: *$s_1(t)$ and $s_3(t)$ are superpositions of $s_2(t)$ and $s_4(t)$ and these latter two are actually orthogonal on $(0, T_b)$. So we just normalize those two to get*

$$\phi_1(t) = \sqrt{2/T_b}\cos\omega_c t$$
$$\phi_2(t) = \sqrt{2/T_b}\cos 2\omega_c t$$

(b) Rewrite the $s_i(t)$ in terms of your orthormal set.

SOLUTION: *We have:*

$$s_1(t) = \tfrac{1}{\sqrt{2}}(\phi_1(t) + \phi_2(t))$$
$$s_2(t) = \tfrac{1}{\sqrt{2}}\phi_1(t)$$
$$s_3(t) = \tfrac{1}{\sqrt{2}}(\phi_1(t) - \phi_2(t))$$
$$s_4(t) = \tfrac{1}{\sqrt{2}}\phi_2(t)$$

or

$$\mathbf{s}_1 = (1, 1)/\sqrt{2/T_b}$$
$$\mathbf{s}_2 = (1, 0)/\sqrt{2/T_b}$$
$$\mathbf{s}_3 = (1, -1)/\sqrt{2/T_b}$$
$$\mathbf{s}_4 = (0, 1)/\sqrt{2/T_b}$$

(c) Assume zero mean white Gaussian noise, $w(t)$, is added to any $s_i(t)$ sent over the channel so that $x(t) = s_i(t) + w(t)$ is received. The noise has spectral height $N_0/2$.

What is the highest probability of error over all the signals? Find an expression for this error.

SOLUTION: *The maximum probability of error depends only the minimium distance between signals. The distance squared between* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *is* $T_b/2$ *as is the distance squared between* $\mathbf{s}_2$ *and* $\mathbf{s}_3$ *and* $\mathbf{s}_1$ *and* $\mathbf{s}_4$.

The noise variance is $N_0/2$ *as usual so the maximum probability of error is*

$$P_e = \frac{1}{2} erfc(\sqrt{\frac{d^2}{4N_0}}) = \frac{1}{2} erfc(\sqrt{\frac{T_b}{8N_0}})$$

(d) Suppose Fly-By-Night Industries offers Cora a new and improved set of signals:

$$\begin{aligned} r_1(t) &= g(t) \\ r_2(t) &= g(t) + h(t) \\ r_3(t) &= g(t) - h(t) \\ r_4(t) &= h(t) \end{aligned}$$

where $g(t)$ and $h(t)$ are proprietary orthogonal waveforms specially designed by FBNI (and they won't actually tell her what they are). The energies in $g(t)$ and $h(t)$ over a bit interval are the same and equal to $T_b/2$. Cora asks whether the probabilities of error are better than he signal set and the company rep smiles as if to say "what do you think?"

Assuming equally likely messages $i = 1, 2, 3, 4$ as before, will this set have better worse or the same probabilities of error compared to Cora's original set?

SOLUTION: *The FBNI signal set is exactly the same constellation as Cora's set when you look in signal space. Therefore their performance, from a* P_e *point of view must be identical. Of course, they might come in prettier colors.... :)*

2. **Optimal Detection with a Twist:**

Here we consider what happens to the decision regions when the noise process does not have the same variance in all orthogonal components.

Consider two signals $s_1(t) = \sqrt{2/T}\cos\omega_c t$ and $s_2(t) = \sqrt{2/T}\sin\omega_c t$ on the interval $(0, T)$ where $T = 2\pi/\omega_c$. These signals are rendered in a signal space with orthonormal components $\phi_1(t) = \sqrt{2/T}\cos\omega_c t$ and $\phi_2(t) = \sqrt{2/T}\sin\omega_c t$.

When these signals are sent over the air, they are corrupted by noise. In signal space we then have $\mathbf{x} = \mathbf{s}_i + \mathbf{w}$ where $\mathbf{s}_i$ is the usual signal space representation of the signal and $\mathbf{w}$ is the noise vector (w_1, w_2). Both w_1 and w_2 are zero mean and Gauusian, BUT have given variances σ_1^2 and σ_2^2 respectively. That is, THE NOISE VARIANCES FOR EACH COMPONENT ARE NOT EQUAL!

(a) Plot $s_1(t)$ and $s_2(t)$ in signal space and provide expressions for the correponding vectors $\mathbf{s}_1$ and $\mathbf{s}_2$.

SOLUTION: *The signal points are exactly the orthonormal waveforms so* $s_1 = (1, 0)$ *and* $s_2 = (0, 1)$.

(b) Write down the proper expressions for the probability distributions $f(\mathbf{x}|\mathbf{s}_1)$ and $f(\mathbf{x}|\mathbf{s}_2)$, $\mathbf{x} = (x_1, x_2)$, the outputs of the correlators in $\phi_1(t)$ and $\phi_2(t)$ respectively.

HINT: Remember we showed that the outputs of the correlators were independent random variables given $\mathbf{s}_i$.

SOLUTION:

Noise is different in different rails of the correlator so

$$f(\mathbf{x}|\mathbf{s}_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}}\frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-(x_1-1)^2/2\sigma_1^2}e^{-x_1^2/2\sigma_2^2} = \frac{1}{2\pi\sigma_1\sigma_2}e^{-(x_1-1)^2/2\sigma_1^2-x_2^2/2\sigma_2^2}$$

$$f(\mathbf{x}|\mathbf{s}_2) = \frac{1}{2\pi\sigma_1\sigma_2}e^{-(x_2-1)^2/2\sigma_2^2-x_1^2/2\sigma_1^2}$$

(c) If $\mathbf{s}_1$ is sent with probability p_1 and $\mathbf{s}_2$ with probability $1-p_1$, write down the likelihood ratio test which describes the minimum probability of error receiver.

SOLUTION:

$$\frac{f(\mathbf{x}|\mathbf{s}_1)}{f(\mathbf{x}|\mathbf{s}_2)} \underset{\textit{say 2}}{\overset{\textit{say 1}}{\gtrless}} \frac{1-p_1}{p_1}$$

(d) Find an expression which relates the x_i to describe the decision region boundary relative the signal points $\mathbf{s}_1$ and $\mathbf{s}_2$. Simplify as much as possible and sketch your decision boundary for $\sigma_1 = 1$, $\sigma_2 = 2$ and $p_1/(1-p_1) = e$.

SOLUTION:

Substitute the appropriate distributions and we have

$$\frac{e^{-(x_1-1)^2/2\sigma_1^2}e^{-x_2^2/2\sigma_2^2}}{e^{-(x_2-1)^2/2\sigma_2^2}e^{-x_1^2/2\sigma_1^2}} \underset{\textit{say 2}}{\overset{\textit{say 1}}{\gtrless}} 1/e$$

Take log of both sides

$$-(x_1-1)^2/2\sigma_1^2 - x_2^2/2\sigma_2^2 + (x_2-1)^2/2\sigma_2^2 + x_1^2/2\sigma_1^2 \underset{\textit{say 2}}{\overset{\textit{say 1}}{\gtrless}} -1$$

Expanding and cancelling yields,

$$-(-2x_1+1)/2\sigma_1^2 + (-2x_2+1)/2\sigma_2^2 \underset{\textit{say 2}}{\overset{\textit{say 1}}{\gtrless}} -1$$

So, the decision region boundary is given by

$$(x_1 - \frac{1}{2}) = (x_2 - \frac{1}{2})\frac{\sigma_1^2}{\sigma_2^2} - \sigma_1^2$$

and when $\sigma_2 = 2\sigma_1 = 2$ *we have*

$$(x_1 - \frac{1}{2}) = (x_2 - \frac{1}{2})\frac{1}{4} - 1$$

which is an offset line. Taking the limit as $\sigma_2 \rightarrow \infty$ *we see that the decision line becomes* $x_1 = \frac{1}{2}$*. This result makes sense if* σ_2 *is very large, then you shouldn't pay much attention to the* x_2 *measurement.*