



# RUTGERS

School of Engineering  
Department of Electrical and Computer Engineering

332:322

## Principles of Communications Systems Linear System Assessment

Spring 2011

*There are four questions. You have the class period to answer them and can use whatever resources you have available (except using a “life line” by calling someone on your cell phone or accessing the solutions on the web. Place all answers on the test sheet. This is an assesment exam and will NOT count toward your final grade. It’s only purpose is for me to determine just how brilliant (or woefully incompetent) you are. :) Try to have fun and exercise what Wade (Prof. Trappe) taught you last term. Depending upon the outcome, Wade might make a special guest appearance for applause (or egg-throwing). :)*

1. The Fourier transform of  $x(t)$  is  $X(f)$ . Prove that the Fourier transform of  $x(t - t_0)$  is  $e^{-j2\pi ft_0} X(f)$ .

**SOLUTION:**

$$\begin{aligned} \mathcal{F}\{x(t - t_0)\} &= \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} x(v) e^{-j2\pi f(v+t_0)} dv \\ &= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} x(v) e^{-j2\pi fv} dv \\ &= e^{-j2\pi ft_0} X(f) \end{aligned}$$

2. The energy in a signal  $x(t)$  is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Prove that if  $X(f)$  is the Fourier transform of  $x(t)$ , then we must also have

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

You may use the fact that

$$\int_{-\infty}^{\infty} e^{j2\pi vt} dt = \delta(v)$$

**SOLUTION:**  $|x(t)|^2 = x(t)x^*(t)$ . We have

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

so that

$$x^*(t) = \int_{-\infty}^{\infty} X^*(f) e^{-j2\pi ft} df$$

Thus,

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} X^*(f') e^{-j2\pi f't} df df' dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f) X^*(f') \left[ \int_{-\infty}^{\infty} e^{j2\pi(f-f')t} dt \right] df df' \end{aligned}$$

where we've replaced  $f$  by  $f'$  to distinguish the dummy variable of integration in the integrals for  $x(t)$  and  $x^*(t)$ . Then, using the fact yields

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f) X^*(f') \delta(f - f') df df' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f) X^*(f) df$$

via the sifting property. Recognizing that  $X(f) X^*(f) = |X(f)|^2$  completes the proof.

3. A signal  $x(t)$  of finite energy is applied to a square-law device whose output is defined by  $y(t) = x^2(t)$ . The spectrum,  $X(f)$ , of  $x(t)$  is limited to the frequency interval  $-W \leq f \leq W$ . Show that the spectrum,  $Y(f)$ , of  $y(t)$  is limited to  $-2W \leq f \leq 2W$ .

**SOLUTION:**

$$y(t) = x^2(t) = x(t)x(t)$$

Since multiplication in time domain corresponds to convolution in frequency domain, we may express the Fourier transform of  $y(t)$  as

$$Y(f) = \int_{-\infty}^{\infty} X(\lambda) X(f - \lambda) d\lambda$$

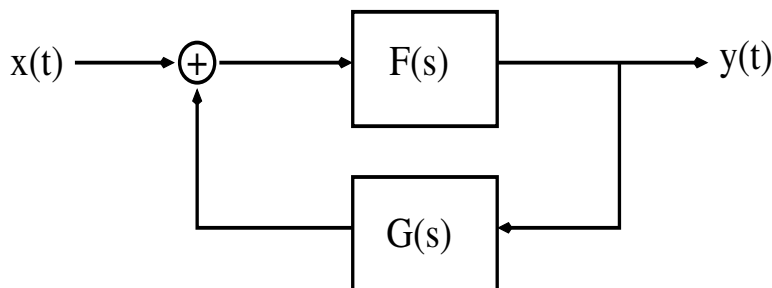
where  $X(f)$  is the Fourier transform of  $x(t)$ . However  $X(f)$  is zero for  $|f| > W$ . Hence,

$$Y(f) = \int_{-W}^W X(\lambda) X(f - \lambda) d\lambda$$

In this integral we note that  $X(f - \lambda)$  is limited to  $-W \leq f - \lambda \leq W$ . When  $\lambda = -W$ , we find that  $-2W \leq f \leq 0$ . When  $\lambda = W$ ,  $0 \leq f \leq 2W$ . Accordingly the Fourier transform  $Y(f)$  is limited to the frequency interval  $-2W \leq f \leq 2W$ .

4. Show that the overall system function  $H(s)$  for the feedback system in FIGURE 1 is given by  $H(s) = \frac{F(s)}{1 - F(s)G(s)}$ .

Figure 1: Feedback system for problem 4.



**SOLUTION:**

$$W(s) = X(s) + G(s).Y(s), Y(s) = F(s).W(s) = F(s)[X(s) + G(s).Y(s)]$$

*Re-arranging the previous equation,*

$$F(s).X(s) = Y(s)[1 - F(s).G(s)]$$

*System funtion*

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(s)}{1 - F(s).G(s)}$$