RUTGERS

School of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2008**

Final Examination

There are 4 questions. You have three hours to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Signal Space 101:** You are given two signals defined on $(-\frac{1}{2}, \frac{1}{2})$: $\phi_1(t) = 1$ and $\phi_2(t) = \alpha t$, where α is a constant.

   (a) *(10 points)* Show that $\phi_1(t)$ and $\phi_2(t)$ are orthogonal.
   SOLUTION:
   $$\int_{-0.5}^{0.5} 1 \cdot \alpha t dt = 0$$
   They're orthogonal

   (b) *(10 points)* Does $\phi_1(t)$ have unit energy on $(-\frac{1}{2}, \frac{1}{2})$? For what value of α does $\phi_2(t)$ have unit energy on $(-\frac{1}{2}, \frac{1}{2})$?
   SOLUTION:
   $$\int_{-0.5}^{0.5} 1^2 dt = 1$$
   $\phi_1(t)$ has unit energy
   $$\int_{-0.5}^{0.5} (\alpha t)^2 dt = \alpha^2 \left. \frac{t^3}{3} \right|_{-0.5}^{0.5} = \alpha^2/12$$
   So, $\alpha = 2\sqrt{3}$ for $\phi_2(t)$ to have unit energy.

   (c) *(10 points)* Let $f(t) = t^2 - 1$. Assume $\alpha = 2\sqrt{3}$ and let $g(t) = a\phi_1(t) + b\phi_2(t)$. Please find the a and b which minimize
   $$\int_{-\frac{1}{2}}^{\frac{1}{2}} (f(t) - g(t))^2 dt$$
   SOLUTION: *This is projection in disguise.*
   $$a = \int_{-0.5}^{0.5} 1 \cdot (t^2 - 1) dt = \frac{1}{12} - 1 = -11/12$$
   and
   $$b = \int_{-0.5}^{0.5} \alpha t \cdot (t^2 - 1) dt = 0$$

That is, if you made the argument that projection minimizes the distance between the projected point and the actual vector, you got full credit.

However, let's do this the long way because we're gnerds and love it. We take partials w.r.t. a and b

$$\frac{\partial}{\partial a}\int_{-0.5}^{0.5}(f(t)-g(t))^2dt = -2\int_{-0.5}^{0.5}(f(t)-g(t))\phi_1(t)dt = -2\left[\int_{-0.5}^{0.5} f(t)\phi_1(t)dt - a\right] = 0$$

Which leads to $a = \langle f(t), \phi_1(t)\rangle$. *Likewise*

$$\frac{\partial}{\partial b}\int_{-0.5}^{0.5}(f(t)-g(t))^2dt = -2\int_{-0.5}^{0.5}(f(t)-g(t))\phi_2(t)dt = -2\left[\int_{-0.5}^{0.5} f(t)\phi_2(t)dt - b\right] = 0$$

and $b = \langle f(t), \phi_2(t)\rangle$. *And just to be extra special safe (remember the warning about minimizing functions of more than on variable?) we take the second partials.*

$$\frac{\partial^2}{\partial a^2}\int_{-0.5}^{0.5}(f(t)-g(t))^2dt = \frac{\partial^2}{\partial b^2}\int_{-0.5}^{0.5}(f(t)-g(t))^2dt = 2$$

and

$$\frac{\partial^2}{\partial a\partial b}\int_{-0.5}^{0.5}(f(t)-g(t))^2dt = 0$$

so the Jacobian has all positive eigenvalues (zero off-diagonal elements and positive diagonal elements) and we have our convex minimization!

(d) *(10 points)* Plot $g(t)$ as a point in the signal space represented by $\phi_1(t)$ and $\phi_2(t)$.

SOLUTION: *The simple-minded answer* (a,b) *got you partial credit, but* $(-11/12, 0)$ *was the final answer.*

(e) *(10 points)* Let $h(t) = f(t) - g(t)$. Is $h(t)$ orthogonal to $\phi_1(t)$? How about to $\phi_2(t)$?

SOLUTION: *Well, you already know the answer to this and could have nailed this by saying that if you subtract off the projections onto orthonormal components, what's left must be orthogonal to those orthogonal components! However, going the long way we've got* $h(t) = (t^2-1) + 11/12 = t^2 - 1/12$.

$$\int_{-0.5}^{0.5}(t^2 - 1/12)\cdot 1 dt = 1/12 - 1/12 = 0$$

Surprise? NOT! Likewise

$$\int_{-0.5}^{0.5}(t^2 - 1/12)\alpha t dt = \alpha\int_{-0.5}^{0.5}(t^3 - t/12)dt = 0$$

because you're integrating odd functions over a symmetric $\pm$ *interval. Surprise? NOT! And dat be dat!*

2. *(50 points)* **Cora Sees the Light:** Late at night, Cora lies in wait for her arch nemisis Marty the Squirrel. She's set up a tube through which Marty must run to forage. At the end of the tube is a remote-controlled mine and Cora holds the transmitter. When an animal runs through the tube, it glows. If it is Marty running through the tube, the glow duration is a

random variable G with exponential distribution $f_{G|M}(g|M) = e^{-g}u(g)$. However Cora's cat, Cassandra, is also about at night. Cassandra is rather fat so her glow duration is distributed as $f_{G|C}(g|C) = \frac{1}{2}e^{-g/2}u(g)$. Cora measures the glow duration and has to make a decision about tripping the mine. Marty or Cassandra traversing the tube are assumed to be equally likely events.

(a) *(20 points)* Derive a decision rule on g that minimizes the probability Cora blows up Cassandra instead of Marty, or misses Marty when he is there.

SOLUTION: *We need a likelihood ratio test to minimize probability of error. Since for both distributions,* $g \geq 0$ *we can write*

$$\frac{\frac{1}{2}e^{-g/2}}{e^{-g}} = \frac{1}{2}e^{g/2} \underset{Marty}{\overset{Cassandra}{\gtrless}} 1$$

We can take the log of both sides and rewrite this as

$$g \underset{Marty}{\overset{Cassandra}{\gtrless}} 2\log 2$$

(b) *(20 points)* Unbeknownst to Cora, Marty places catnip laced with caffeine at the tube entrance. Cassandra will eat the catnip and her glow time distribution will become $f_{G|C}(g|C) = 2e^{-2g}u(g)$. If Cora uses the decision rule of the previous part, what is the probability she blows up Cassandra?

SOLUTION: *The probability Cassandra gets blown up is the probability she is the one in the tube (1/2) times the probability that* $G \leq 2\log 2$ *given the catnip distribution. That is, Cora will pull the trigger if* $G \leq 2\log 2$ *because that's the optimum decision rule based on her previous assumptions.*

So,

$$P_{C_{kaboom}} = \frac{1}{2}\int_0^{2\log 2} 2e^{-2g}dg = \frac{1}{2}(1 - e^{-2\cdot 2\log 2}) = \frac{1}{2}(1 - 2^{-4}) = \frac{15}{32}$$

(c) *(10 points)* Cora decides Cassandra-be-damned and only seeks to maximize the probability she blows up Marty. What's her optimum decision rule? What's the probability Marty's blown up?

SOLUTION: *You could go through the derivation if you wanted, but that's a bit boneheaded. Clearly if she doesn't care, she should pull the trigger whenever the tube glows, no matter what g is. In that case, the probability of getting Marty is* $1/2$.

3. *(50 points)* **Signal Space 102:** Signals $s_1(t)$ and $s_2(t)$ are sent equiprobably over a channel which corrupts them with additive zero mean white Gaussian noise $w(t)$ of spectral height 1. That is, the received signal $r(t) = s_k(t) + w(t)$ where k is either 1 or 2. You are told $s_1(t) = u(t)$ and $s_2(t) = (u(t) - 2u(t - 0.5))$ and that both signals exist only on $(0, 1)$.

(a) *(10 points)* Sketch the signals $s_1(t)$ and $s_2(t)$ and derive an appropriate signal space for them. What is the dimension of your signal space? (EASY)

SOLUTION: *The signals are orthogonal on* $(0,1)$. *Furthermore, both are unit energy. So, just use* $\phi_1(t) = s_1(t)$ *and* $\phi_2(t) = s_2(t)$. *The dimension is obviously* 2.

(b) *(10 points)* Plot the signal points corresponding to $s_1(t)$ and $s_2(t)$ in this signal space. (EASY)

SOLUTION: *The signal point for* $s_1(t)$ *is just* $(1,0)$ *while that for* $s_2(t)$ *is* $(0,1)$

(c) *(10 points)* Assuming a correlator receiver based on your signal space basis functions, derive a minimum probability of error decision rule and sketch the decision region in your signal space. (EASY)

SOLUTION: *We already derived that the minimum probability of error rule for signals in Gaussian noise is distance-based decoding. You choose the signal closest to the point you observe. Thus, the decision boundary is the line which bisects the line which passes through the two signal points – in this case a* 45^o *line which passes through the origin.*

(d) *(10 points)* Derive explicit expressions for $f_{R_1R_2|s_2}(r_1,r_2|s_1)$ and $f_{R_1R_2|s_1}(r_1,r_2|s_2)$. (NOT HARD)

SOLUTION:

$$f_{R_1R_2|s_1}(r_1,r_2|s_1) = \frac{1}{\sqrt{2\pi}}e^{-(r_1-1)^2/2}\frac{1}{\sqrt{2\pi}}e^{-r_2^2/2} = \frac{1}{2\pi}e^{-((r_1-1)^2+r_2^2)/2}$$

Likewise

$$f_{R_1R_2|s_2}(r_1,r_2|s_2) = \frac{1}{2\pi}e^{-((r_2-1)^2+r_1^2)/2}$$

(e) *(10 points)* If we set $s_2(t) = 0$ how does the decision rule change? Discuss quantitatively (or qualitatively for half credit) whether the probability of error increases or decreases compared to $s_2(t) = (u(t) - 2u(t-0.5))$.

SOLUTION: *The decision boundary is still the line which bisects the line passing through the signal points. However, the new line connecting the points is the* ϕ_1 *axis, so the decision rule is* $R > 1/2$ *say* $s_1()$, *otherwise say* $s_2()$.

The distance between the original two signals was $\sqrt{2}$. *The new distance is* $1 < \sqrt{2}$ *So, for the same noise variance, we've got more distribution "skirt" overlap and higher probability of error.*

Quantitatively, we can first do a translation of the signal points – i.e., shifting things won't change the region of integration for the probability of error. That is

$$f'_{R_1R_2|s_1}(r_1,r_2|s_1) = \frac{1}{2\pi}e^{-((r_1-1)^2+(r_2-1)^2)/2}$$

and

$$f'_{R_1R_2|s_2}(r_1,r_2|s_2) = \frac{1}{2\pi}e^{-(r_2^2+r_1^2)/2}$$

For the original case, the probability of error was

$$P_e = \int_{-\infty}^{\infty}\int_{-\infty}^{r_1}\frac{1}{2\pi}e^{-((r_2-1)^2+r_1^2)/2}dr_2dr_1$$

In the translated coordinate system it becomes

$$P'_e = \int_{-\infty}^{\infty}\int_{-\infty}^{r_1-1} \frac{1}{2\pi} e^{-(r_2^2+r_1^2)/2} dr_2 dr_1$$

But $e^{-(r_2^2+r_1^2)/2}$ *is circularly symmetric about the origin, so we can integrate the probability under the "skirt" whose boundary is ANY line* $\sqrt{2}/2$ *away from the origin. Thus,*

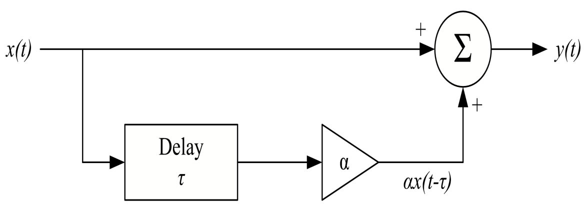
$$P'_e = \int_{-\infty}^{\infty}\int_{\frac{\sqrt{2}}{2}}^{\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy = \int_{\frac{\sqrt{2}}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

which is clearly less than

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r_1/2} dr_1$$

the probability of error for the case when $s_2(t) = 0$.

4. *(50 points)* **Multipath Channel Equalization:** In wireless channels, radio (or sound) waves bounce all over the place and recombine at the receiver causing echoes which can degrade the signal. Your job is to derive an equalizer which takes the incoming signals and removes the echoes as well as possible. So, consider the channel and equalization filter shown in FIGURE 1



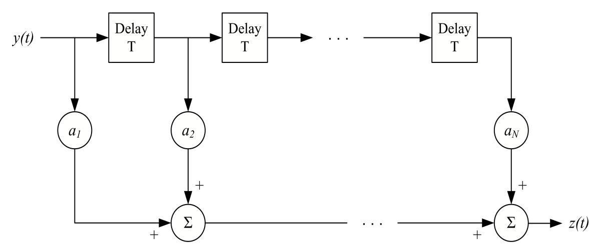


Figure 1: LEFT: Channel model. RIGHT: Equalization filter

$\tau \in \Re^+$ and $\alpha \in \Re$.

(a) *(10 points)* $x(t)$ is the channel input. What is the channel output $y(t)$ in terms of $x(t)$?
SOLUTION: $y(t) = x(t) + \alpha x(t-\tau)$

(b) *(10 points)* What is the impulse response of the channel, $h(t)$? What is the frequency response of the channel, $H(f)$?
SOLUTION: $h(t) = \delta(t) + \alpha\delta(t-\tau)$ *so* $H(f) = 1 + \alpha e^{-j2\pi f\tau}$

(c) *(10 points)* Show that

$$z(t) = \sum_{k=1}^{N} a_k y(t-(k-1)T)$$

What is the impulse response, $g(t)$ of this transversal filter? **HINT:** Don't overthink this.

SOLUTION: *Just walk through the signal flow of the diagram. The impulse response (when* $y(t) = \delta(t)$ *is thus*

$$g(t) = \sum_{k=1}^{N} a_k \delta(t-(k-1)T)$$

– this was an easy 10.

(d) *(10 points)* What is $Z(f)$ the Fourier transform of $z(t)$? What is $G(f)$ the Fourier transform of $g(t)$?

SOLUTION:

$$Z(f) = \sum_{k=1}^{N} a_k Y(f) e^{-j2\pi f(k-1)T}$$

and

$$G(f) = \sum_{k=1}^{N} a_k e^{-j2\pi f(k-1)T}$$

(e) *(10 points)* Assume $\tau = T$. What is the overall impulse response $q(t)$ of the channel filter followed by the transversal filter? Assume $|\alpha| < 1$ what values should be chosen for the a_k, $k = 1, 2, \cdots N$ so that $q(t)$ is as close to a perfect channel as possible; i.e., $q(t) \approx \delta(t)$. Put more formally, we seek

$$\{a_k^*\} \arg\min_{\{a_k\}} \int_0^{kT} |q(t) - \delta(t)| dt$$

HINT: Try a hueristic approach before going formal. If you go formal, remember that $d|x|/dx = sgn(x)$ except at $x = 0$ where it's defined as 0.

SOLUTION:

$$(g * h)(t) = q(t) = \left[\sum_{k=1}^{N} a_k \delta((k-1)T) \right] * [1 + \alpha\delta(t - T)]$$

Simplifying we have

$$q(t) = a_1\delta(t) + \sum_{k=1}^{N-1} (a_{k+1} + \alpha a_k)\delta(t - kT) + \alpha a_N \delta(t - NT)$$

Then,

$$q(t) - \delta(t) = (a_1 - 1)\delta(t) + \sum_{k=1}^{N-1} (a_{k+1} + \alpha a_k)\delta(t - kT) + \alpha a_N \delta(t - NT)$$

We thus seek

$$\{a_k^*\} \arg\min_{\{a_k\}} |a_1 - 1| + \sum_{k=1}^{N-1} |a_{k+1} + \alpha a_k| + |\alpha a_N|$$

If we set $a_1 = 1$ and then $a_{k+1} = -\alpha a_k$ we are left with $|\alpha a_N| = |\alpha|^N$ which decreases geometrically in N since $|\alpha| < 1$.

Of course, that was not a formal minimization. To do this formally, we need to differentiate with respect to the variables. $d|x|/dx = sgn(x)$ except at $x = 0$ where it's defined as 0.

$$\frac{\partial}{\partial a_1} |a_1 - 1| + \sum_{k=1}^{N-1} |a_{k+1} + \alpha a_k| + |\alpha a_N| = sgn(a_1 - 1) + \alpha sgn(a_2 + \alpha a_1)$$

Clearly, $a_1 = 1$ *and* $a_2 - \alpha$ *sets this to zero. Likewise,*

$$\frac{\partial}{\partial a_k}|a_1 - 1| + \sum_{k=1}^{N-1} |a_{k+1} + \alpha a_k| + |\alpha a_N| = \alpha sgn(a_{k+1} + \alpha a_k) + sgn(a_k + \alpha a_{k-1})$$

for $1 < k < N$*. Setting* $a_{k+1} = -\alpha a_k$ *works here too. Finally*

$$\frac{\partial}{\partial a_N}|a_1 - 1| + \sum_{k=1}^{N-1} |a_{k+1} + \alpha a_k| + |\alpha a_N| = sgn(a_N + \alpha a_{N-1}) + |\alpha| sgn(a_N)$$

and we're stuck – no way to force this to zero, so we do the next best thing and minimize the first term by setting $a_N = -\alpha a_{N-1}$.