THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2006**

Final Exam

There are FIVE questions. You have the three hours to answer them. Read the WHOLE EXAM before doing the problems. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Amplitude Modulation:** A signal $r(t) = m(t)(\cos 200000\pi t + \sin 200000\pi t)$ is sent over the air and captured by a receiver. You may assume the program material $m(t)$ is band limited to $\pm$1kHz and is always non-negative.

   (a) *(25 points)* Please sketch a block diagram for ANY receiver which will recover $m(t)$.

   SOLUTION: *A variety of receivers will work here. Synchronous detection using almost any linear combination of* $\cos 200000\pi t$ $\sin 200000\pi t$ *will work (multiplication and LPF). The only one which will NOT work is* $(\cos 200000\pi t - \sin 200000\pi t)$ *because you're then multiplying by a sinuoid that's* $\pi/2$ *out of phase with the carrier (like multiplying sin by cos or* vice versa*). One could also use an envelope detector since we can be assured that* $m(t) \geq 0\ \forall t$.

   (b) *(25 points)* Let

   $$c(t) = \sum_k (u(t - kT) - u(t - (k + 1/2)T)$$

   and suppose we can synthesize $c(t - \tau)$ at our receiver for any $\tau \in \Re$ and $T \in \Re^+$. Please show how to build an envelope detector WITHOUT DIODES for detection of $m(t)$ using our generated signal $c(t - \tau)$ and a multiplier followed by a capacitor and resistor. A sketch would be most effective, but be sure to justify your answer.

   SOLUTION: *An envelope detector simply rectifies the signal and then follows the peaks with a resistor/capacitor parallel circuit. We notice that* $(\cos 200000\pi t + \sin 200000\pi t)$ *is positive for* $t \in 10^{-5}(-\frac{1}{8} + n, \frac{3}{8} + n)$ *for all integer n. Since* $m(t)$ *is non-negative, all we need to do is form* $c(t + 10^{-5}/8)$ *with* $T = 10^{-5}$ *and multiply it by* $r(t)$ *to obtain*

   $$r^+(t) = r(t) = m(t)(\cos 200000\pi t + \sin 200000\pi t)^+$$

   where the superscript + notation means zero if negative, argument if positive. This is a half-wave rectification of $r(t)$. $r^+(t)$ *can then be applied to the parallel capacitive/resistive part of a peak detector to obtain* $m(t)$.

   The RC time constant needs to be fast enough to track $m(t)$ *but slow enough to suppress the fast variation of the rectified carrier. The carrier frequency is* 10^5*Hz and the frequency content of* $m(t)$ *is confined to* $\pm$*1kHz. So having an RC time constant of* 10^{-4} *should suffice.* $R = 1MOhm$ *and* $C = 100pF$ *will work just fine (about the impedance and capacitance of a set of electrostatic headphones, by the way).*

2. *(50 points)* **Frequency Modulation:** Program material $m(t)$, bandlimited to $\pm W$, is transmitted using frequency modulation as in

$$r(t) = A\cos\left(2\pi f_c t + \beta\int_0^t m(\tau)d\tau\right)$$

where A is a constant, $f_c \gg W$ is the carrier frequency and β is a modulation index.

(a) *(25 points)* Provide an explicit approximate analytic expression for $r(t)$ when it is assumed $|\beta| \ll 1$ (narrowband FM). Can $m(t)$ can be obtained via synchronous amplitude demodulation using $\cos 2\pi f_c t$? Why/why not?

SOLUTION:

$$r(t) \approx A\cos(2\pi f_c t) + A\beta m(t)\sin(2\pi f_c t)$$

using the trigonometric expansion for $\cos(a+b)$ *and the approximation* $\sin(x) \approx x$ *for* x *small.*

If we put this $r(t)$ *into synchronous demodulator with* $\cos(2\pi f_c t)$*, the term in* $\beta m(t)$ *will disappear after the low pass filter. So no,* $m(t)$ *can't be recovered via synchronous demodulation using* $\cos()$.

(b) *(25 points)* Provide a carefully labeled block diagram of a phase locked loop FM receiver which extracts $m(t)$ from $r(t)$. Your diagram should include a multiplier, a low pass filter, a scalar multiplication (amplifier) and a voltage controlled oscillator. Should the amplifier be an inverting or a non-inverting amplifier? What is the minimum bandwidth of the low pass filter? You must justify your answers.

SOLUTION: *You can find a picture in the text, in our notes and in the problem sets. You multiply the incoming signal by the output of a VCO. The resulting signal is filtered and then inverted and amplified (a LOT). This signal, the output of the PLL, is fed back to the input of the VCO which integrates its input and uses the result as the argument of* $\sin()$. *The idea behind the phase locked loop is NEGATIVE FEEDBACK. If the output of the VCO and the incoming signal are* $\pi/2$ *out of phase, the output of the low pass filter is zero – which means that we've achieved "lock". Otherwise, a positive phase "miss" will be corrected by a negative shift in phase at the input of the VCO and a negative miss will be corrected by a positive shift.*

3. *(50 points)* **Sampling Theory, Signal Space and Parseval:** A signal $m(t)$ bandlimited to $\pm W$ can be represented by its samples $m(kT)$ where $T \leq \frac{1}{2W}$. This so-called Nyquist Sampling Theorem is most easily proven by considering impulsive sampling $\sum_k m(kT)\delta(t-kT)$ and applying this signal to an ideal low pass filter with bandwidth $\pm W$ to recover $m(t)$.

(a) *(10 points)* Show that an ideal low pass filter with bandwidth $\pm W$ has an impulse response **proportional to**

$$h(t) = \frac{1}{\sqrt{2W}}\frac{\sin 2\pi W t}{\pi t}$$

SOLUTION: $H(f) = u(f+W) - u(f-W)$ *so*

$$h(t) = \int_{-W}^{W} e^{j2\pi f t}df = \frac{\sin 2\pi W t}{\pi t}$$

(b) *(20 points)* Please provide an expression for $m(t)$ in terms of its samples $\{m(kT)\}$ and $h(t)$.

SOLUTION:

$$m(t) = \sqrt{2W}\sum_k m(kT)h(t-kT)$$

(c) *(20 points)* Please verify that the $\{h(t-kT)\}$ are orthonormal waveforms. You might find Parseval's relation useful:

$$\int_{-\infty}^{\infty} x(t)y(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

where $x(t)$ and $y(t)$ are real signals and $X(f)$ and $Y(f)$ are their respective Fourier transforms.

SOLUTION:

$$< h(t-kT), h(t-nT) >= \frac{1}{2W}\int_{-\infty}^{\infty} h(t-kT)h(t-nT)dt = \frac{1}{2W}\int_{-\infty}^{\infty} e^{-j2\pi kTf}H(f)e^{j2\pi nTf}H^*(f)df$$

Simplifying

$$< h(t-kT), h(t-nT) >= \frac{1}{2W}\int_{-\infty}^{\infty} |H(f)|^2 e^{-j2\pi(k-n)Tf}df = \frac{1}{2W}\int_{-W}^{W} e^{-j2\pi(k-n)Tf}df$$

Since $T = 1/2W$ the integral limits are integral numbers of a period of the complex exponential argument. So for $n \neq k$ we get zero. And when $n = k$ the integal is $2W$ so the expression evaluates to 1. Thus

$$< h(t-kT), h(t-nT) >= \delta_{kn}$$

and are thus orthonormal.

4. *(50 points)* **Digital Detection:** You are given two signals on the interval $(-1,1)$: $s_0(t) = u(t+1) - u(t-1)$ and $s_1(t) = t$. These signals are to be used to send information bits with $s_0(t)$ corresponding to a zero and $s_1(t)$ to a one. Thus

$$r(t) = s_i(t) + n(t)$$

where $i = \{0,1\}$ and $n(t)$ is zero mean white Gaussian noise with spectral height σ^2.

(a) *(25 points)* Find a signal space with two orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ in which both $s_1(t)$ and $s_0(t)$ can be represented. Label the points in your signals space which correspond to $s_1(t)$ and $s_0(t)$.

SOLUTION: *$s_0(t)$ and $s_1(t)$ are clearly orthogonal since their product, t, is an odd function which when integrated from -1 to 1 must be zero. So all we need to do is normalize them. The energy in $s_0(t)$ is 2 so we have*

$$\phi_1(t) = \frac{1}{\sqrt{2}}s_0(t)$$

Likewise, the energy in $s_1(t)$ is

$$\int_{-1}^{1} t^2 dt = \frac{2}{3}$$

so we have

$$\phi_2(t) = \sqrt{\frac{3}{2}} s_1(t)$$

Therefore, the point corresponding to $s_0(t)$ *in this signal space is* $\mathbf{s}_0 = (\sqrt{2}, 0)$ *and the point corresponding to* $s_1(t)$ *is* $\mathbf{s}_1 = (0, \sqrt{2/3})$.

(b) *(25 points)* Please provide a block diagram for a minimum probability of error receiver assuming that sending a one or zero is equally likely. Sketch the decision region of your receiver in the signal space.

SOLUTION: *We were given the signal is embedded in white gaussian zero mean noise and that the signals are equally likely. Thus, we have Gaussian "clouds" around the signal points and minimum distance decoding applies. The decision line is the perpendicular bisector of a line drawn between* $\mathbf{s}_0$ *and* $\mathbf{s}_1$.

5. *(50 points)* **Cora Does a Double Take:** Cora the communications engineer has been challenged to a detection duel by her arch nemisis, Marty the Squirrel. Marty gets to choose a symbol b and transmit it. Cora has access to one of two received signals $r_1(t)$ and $r_2(t)$ where

$$r_1(t) = bs(t) + n(t)$$

$$r_2(t) = 0.1bs(t) + n(t)$$

The signal $s(t)$ is a unit energy signature waveform, $b = \pm 1$ equiprobably and $n(t)$ is a zero mean Gaussian white noise signal with spectral height $N_0/2$. Cora has to guess whether -1 or $+1$ was sent. If she guesses right, Cora gets to use her flamethrower on Marty's bushy tail. If she guesses wrong, Marty gets to tap dance on her face with his hobnail squirrel boots.

(a) *(25 points)* Assuming matched filter detection, if Cora has a choice of using either $r_1(t)$ or $r_2(t)$, which should she use to minimize her chances of facial dents? You must justify your answer.

SOLUTION: *The signal to noise ratio of* $r_1(t)$ *is* $1/(N_0/2)$ *whereas it is* $0.01/(N_0/2)$ *for* $r_2(t)$. *Since increased signal to noise ratio reduces probability of error, if Cora wants to be right (not have Marty dance on her face) she should use choose* $r_1(t)$ *to decode.*

(b) *(25 points)* Now suppose Cora has access to both $r_1(t)$ and $r_2(t)$. Please provide a block diagram of a receiver which maximizes the probability of a flaming tail. What is this probability?

SOLUTION: *This question is the reason I ask everyone to GET SOME SLEEP before they take my exams. If Cora has access to BOTH* $r_1(t)$ *and* $r_2(t)$ *then she also has two copies of the noise* $n(t)$. *So, she can form*

$$r_1(t) - r_2(t) = 0.9bs(t)$$

and COMPLETELY SUBSTRACT THE NOISE. With no noise, there is no uncertainty, so the probability of a flaming tail is identically 1.