THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2004**

Final Examination

There are FIVE questions. You have three hours to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(35 points)* **Continuous Modulation:** For each of the following parts, $m(t)$ is analog program material, band limited to $\pm W$ Hertz with $|m(t)| \leq 1$. The carrier frequency is $\omega_c \gg 2\pi W$ and the signal at the receiver is $r(t)$.

   (a) *(15 points)* If $r(t) = m(t)\cos\omega_c t$, carefully sketch a block diagram for a receiver which can successfully recover $m(t)$.

   SOLUTION: *You cannot simply use an envelope detector directly since $m(t)$ could be negative and the envelope of $r(t)$ is $|m(t)|$ (5 points for knowing this was amplitude modulation). If the carrier frequency and phase are known exactly at the receiver then you could use synchronous detection where you apply a low pass filter to $r(t)\cos\omega_c t$ (10 points for understanding synchronous detection). However, the frequency and phase will in practice be unknown at the receiver so you must use a phase locked loop to first recover the carrier $\cos\omega_c t$ and then do synchronous demodulation (15 points).*

   (b) *(10 points)* You are given a wideband FM signal $r(t) = \cos\left(\omega_c t + k_f \int_0^t m(\tau)d\tau\right)$. Assuming $0 < k_f < \omega_c$, carefully sketch a block diagram for a receiver which has as one of its elements an envelope detector.

   SOLUTION: *Simply differentiate to get $-\left(\omega_c + k_f m(t)\right)\sin\left(\omega_c t + k_f \int_0^t m(\tau)d\tau\right)$, invert and pass the result through an envelope detector.*

   (c) *(10 points)* If $m(t) = \cos 2000\pi t$ and $k_f = 20$, what is the approximate bandwidth of the modulated signal $r(t)$.

   SOLUTION: $\int_0^t \cos 2000\pi\tau d\tau = \frac{1}{2000\pi}\sin 2000\pi t$, *so the modulation index is* $\beta = k_f/(2000\pi) \ll 1$. *So, this is narrowband FM and the bandwidth is approximately that of $m(t)$, or 1000 Hz to either side of the carrier $f_c = \frac{\omega_c}{2\pi}$.*

2. *(30 points)* **Quantization:** A signal has amplitude PDF $f_X(x)$ and we wish to design a quantizer $Q(x)$ with N levels which minimizes the mean square error between X and $Q(x)$.

   PROVE: if $f_X(x)$ is uniform on $\pm A$ where A is some constant, equal sized steps between vertical levels is optimal.

   HINT: The solution to the homogeneous difference equation $y(k+2) - 2y(k+1) + y(k) = 0$ is $\alpha + k\beta$ where α and β are constants.

SOLUTION: *We start with Lloyd-Max:*

$$x_k = \frac{q_{k+1} + q_k}{2}$$

$k = 1, 2, \ldots, N-1$ *and*

$$q_k = E\left[X | X \in (x_{k-1}, x_k)\right]$$

Since the distribution $f_X(x)$ *is uniform we then have*

$$q_k = \frac{x_k + x_{k-1}}{2}$$

for $k = 1, 2, \ldots N$.

We then have

$$q_k = \frac{q_{k+1}}{4} + \frac{q_k}{2} + \frac{q_{k-1}}{4}$$

or

$$q_{k+1} - 2q_k + q_{k-1} = 0$$

a difference equation whose solution is

$$q_k = \alpha + \beta k$$

We then note that $q_k - q_{k-1} = \beta$ *a constant, which completes the proof.*

3. *(50 points)* **Stationarity and Gaussian Processes:**

   (a) *(10 points)* Formally define Strict Sense Stationarity (SSS) for a random process, $X(t)$.
   SOLUTION: *A process* $X(t)$ *is SSS if for any set of times* $\{t_k\}$ *and any offset* τ *we have*

   $$f_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots x_N) = f_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_N+\tau)}(x_1, x_2, \ldots x_N)$$

   (b) *(10 points)* Formally define Wide Sense Stationarity (WSS) for a random process, $X(t)$.
   SOLUTION: *A process* $X(t)$ *is wide sense stationary if its mean* $E[X(t)] = \mu$*, a constant and if its autocorrelation* $R_X(t, \tau) = E[X(t)X(\tau)] = R_X(\tau)$. *That is, the autocorrelation should depend only on the time difference between the samples of* $X()$.

   (c) *(15 points)* A zero mean white Gaussian process $X(t)$ with spectral height N_0 is integrated to obtain $Z(t) = \int_0^t X(\tau)d\tau$, $t \geq 0$. What is the mean of $Z(t)$? What is its correlation? Is $Z(t)$ stationary? Why/why not?

   SOLUTION: *The mean is zero (just propagate the expectation inside the integral). The correlation is*

   $$\int_0^{t_1} \int_0^{t_2} E[X(\tau)X(\sigma)]d\tau d\sigma = N_0 \int_0^{t_1} \int_0^{t_2} \delta(\tau - \sigma)d\tau d\sigma = \begin{cases} t_1 N_0 & t_1 \leq t_2 \\ t_2 N_0 & t_1 \geq t_2 \end{cases}$$

   so this process is non-stationary

(d) *(10 points)* What is the average power of the process on the interval $(0,T)$ where the average power of a signal $x(t)$ is defined as

$$\bar{P} = \frac{1}{T}\int_0^T x^2(t)dt$$

SOLUTION: *The average power*

$$E\left[\frac{1}{T}\int_0^T \left(\int_0^t X(\tau)d\tau\right)^2 dt\right] = \frac{1}{T}\int_0^T N_0 t dt = N_0\frac{T}{2}$$

(e) *(5 points)* Is $Z(t)$ in the previous part a Gaussian process? Why/why not?

SOLUTION: *Well, certainly each sample of $Z(t)$ is a Gaussian random variable since it's the sum (integral) of a set of Guassian random variables. Likewise, since each sample of $Z(t)$ is a sum over the same set of jointly gaussian random variables $X(t)$, the $Z(t)$ must also be jointly Gaussian. Therefore, $Z(t)$ is a nonstationary Gaussian process.*

4. *(60 points)* **Cora and the Alien Invasion:** Cora the Communications Engineer has been hired by NASA to investigate the possibility of an imminent alien invasion through measurements taken by the Voyager deep space probe as it pierces the heliosphere (extended solar neighborhood). Through top secret research, NASA has determined that the signal level S follows the following distributions:

$$f_{S|H_0}(s|H_0) = \lambda^2 s e^{-\lambda s}$$

and

$$f_{S|H_1}(s|H_1) = \lambda s e^{-\frac{1}{2}\lambda s^2}$$

where H_1 means the aliens are planning an invasion, and H_0 not. In both cases, $s \geq 0$. Your job is to help Cora design a decision box which takes the measurement S and produces a decision about whether the aliens are invading or not and does so with minimum probability of error.

(a) *(20 points)* Please sketch the two conditional distributions for $\lambda = 1$.

SOLUTION: *See FIGURE 1.*

$g := (s, \lambda) \rightarrow \lambda s e$

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> plot({f(s,1),g(s,1)},s=0.0..5);
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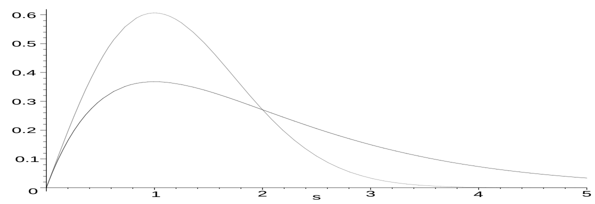


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>
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Figure 1: Plot for Cora problem and $\lambda = 1$

(b) *(20 points)* If the aliens are planning an invasion with probability p, please provide an appropriate likelihood ratio for the decision regions associated with H_0 and H_1.
SOLUTION:

$$\frac{p\, f_{S|H_1}(s|H_1)}{(1-p)\, f_{S|H_0}(s|H_0)} = \frac{p\lambda s e^{-\frac{1}{2}\lambda s^2}}{(1-p)\lambda^2 s e^{-\lambda s}} = \frac{p e^{-\frac{1}{2}\lambda s^2}}{(1-p)\lambda e^{-\lambda s}} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

(c) *(20 points)* Please determine analytic expressions for decision regions for $p = 0.5$. What does your region reduce to if $\lambda = 1$?
SOLUTION: *For $p = 0.5$ we have*

$$\frac{e^{-\frac{1}{2}\lambda s^2}}{\lambda e^{-\lambda s}} = e^{-\frac{1}{2}\lambda s(s-2)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

Since e^{-s^2} goes to zero much faster than e^{-s} and since the integral under each conditional distribution must be one (probability functions), we must have the H_1 conditional distribution larger over some region than the H_0 distribution.
So we seek to solve the equation for values of s where

$$e^{-\frac{1}{2}s(s-2)} = \lambda$$

Taking the log we have

$$-\frac{1}{2}\lambda s(s-2) = \log\lambda$$

so that

$$s^2 - 2s + a = 0$$

where $a = \frac{2}{\lambda}\log\lambda$. Therefore

$$s = 1 \pm \sqrt{1-a}$$

So on the interval $(1-\sqrt{1-a}, 1+\sqrt{1-a})$ we choose H_1.
For $\lambda = 1$ this reduces to $H_1 = \{s | s \in (0,2)\}$

5. *(40 points)* **Signal Space:** A two-dimensional signal space uses basis functions $\phi_1(t) = 1$ and $\phi_2(t) = \sqrt{2}\cos 2\pi t$ on an interval $(0,1)$.

(a) *(10 points)* Please provide a signal space vector representation $\mathbf{s}_k$ for each of the functions $s_1(t) = \cos^2 \pi t$ and $\sin^2 \pi t$ so that

$$s_k(t) = \mathbf{s}_k^\top \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = s_{k1}\phi_1(t) + s_{k2}\phi_2(t)$$

What is the energy in each signal? What is the distance between these two points in signal space?

SOLUTION: $\cos^2 \pi t = \frac{1}{2} + \cos 2\pi t$ *and* $\sin^2 \pi t = \frac{1}{2} - \cos 2\pi t$. *So* $\mathbf{s}_1 = [\frac{1}{2}, \frac{1}{\sqrt{2}}]$ *and* $\mathbf{s}_2 = [\frac{1}{2}, -\frac{1}{\sqrt{2}}]$.

The energy in each is the same and equal to $\frac{3}{4}$

The distance is $\sqrt{2}$.

(b) *(10 points)* On an interval $(0,1)$ one of these signals is sent with equal probability. Zero mean white Gaussian noise of spectral height N_0, $w(t)$, is added to the transmission and a minimum probability of error receiver is used to decode whether signal 1 or signal 2 was sent. Under this scenario, the probability of error is some number P_e

Now, suppose we change the signal design and keep $s_1(t)$ as is but change $s_2(t) = -s_1(t)$. Does the total signal energy change? Does the probability of error go up or down? Why?

SOLUTION: *With white noise, the distance between the signal points matters. Under the replacement, the signal energies stay the same, but the distance increases to* $\sqrt{3}$. *Since the noise variances have not changed, this places the conditional Gaussian distributions which form the likelihood ratio farther apart in signal space and makes the integral under the tails (actually "skirts" since we're in 2-D) of the distributions smaller. So* P_e *goes down.*

(c) *(20 points)* Plot the two pairs of signal points in the same signal space coordinate frame and comment on binary signal design (where you have only two possible signals) under signal energy constraints in a multidimensional signal space. You may guess (with verbal justification) but to receive full credit you must PROVE your assertion.

SOLUTION: *When given an energy budget and only two signals, you should make them antipodal (opposites of each other) to maximize the distance and hence minimize the probability of error.*

Formally, we seek to maximize $d^2 = |\mathbf{s}_1 - \mathbf{s}_2|^2$ *subject to a fixed energy constraint* $E = |\mathbf{s}_1|^2 + |\mathbf{s}_2|$. *First, we rewrite things as*

$$d^2 = (\mathbf{s}_1 - \mathbf{s}_2)^\top (\mathbf{s}_1 - \mathbf{s}_2) = |\mathbf{s}_1|^2 + |\mathbf{s}_2|^2 - 2\mathbf{s}_1^\top \mathbf{s}_2 = E - 2\mathbf{s}_1^\top \mathbf{s}_2$$

Since we want to make d *as large as possible, we need to make* $\mathbf{s}_1^\top \mathbf{s}_2$ *as negative as possible. Well, for fixed length vectors, we already know that the projection is maximized when both vectors are pointing in the same direction. So the dot product will be minized when they're pointing in opposite directions. The only thing left is to determine their amplitude. Let the amplitude of* $\mathbf{s}_1$ *be* x *and that of* $\mathbf{s}_2$ *be* y. $E = x^2 + y^2$ *so we can write* $y = \sqrt{E - x^2}$. *Then we have* $\mathbf{s}_1^\top \mathbf{s}_2 = -xy = -xsqrtE - x^2$ *which is maximized when* $x = \sqrt{E/2}$.

Don't believe me? We can just as easily maximize the square $x^2(E - x^2)$. *Take the first derivative and set to zero:* $2xE - 4x^3 = 0$ *which is satisfied by* $x = \frac{\sqrt{E}}{2}$ *and* $x = 0$.