R, C, and L Elements and their v and i relationships

We deal with three essential elements in circuit analysis:

- Resistance R
- Capacitance C
- Inductance L

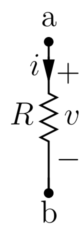
Their v and i relationships are summarized below.

Resistance:

Time domain:

$$v(t) = Ri(t)$$

$$i(t) = \frac{v(t)}{R}$$



Time Domain

Resistance is a *static element* in the sense $v(t)$ versus $i(t)$ relationship is instantaneous. For example, $v(t)$ at time $t = 2$ seconds simply depends only on $i(t)$ at $t = 2$ seconds and nothing else. This implies that the resistance does not know what happened in the past, in other words it does not store any energy unlike other elements C and L as we see soon.

Capacitance:

Time domain:

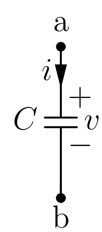
$$q(t) = Cv(t)$$

$$i(t) = \frac{dq}{dt} = C\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{1}{C}i(t)$$

$$\int_{t_o}^{t} \frac{dv}{dt} = v(t) - v(t_o) = \frac{1}{C}\int_{t_o}^{t} i(\tau)d\tau.$$

$$v(t) = v(t_o) + \frac{1}{C}\int_{t_o}^{t} i(\tau)d\tau.$$



Time Domain

Unlike in the case of resistance, for a capacitance the $v(t)$ versus $i(t)$ relationship and vice versa at any time t depends on the past as they involve differentials and integrals. This implies that the capacitance is a *dynamic element.* What happened in the past influences the present behavior. As we shall see soon, capacitance stores energy.

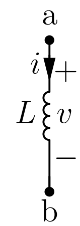
Inductance:

Time domain: $$v(t) = L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}v(t)$$

$$\int_{t_o}^{t} \frac{di}{dt} = i(t) - i(t_o) = \frac{1}{L}\int_{t_o}^{t} v(\tau)d\tau.$$

$$i(t) = i(t_o) + \frac{1}{L}\int_{t_o}^{t} v(\tau)d\tau.$$



Time Domain

Once again, unlike in the case of resistance, for an inductance the $v(t)$ versus $i(t)$ relationship and vice versa at any time t depends on the past as they involve differentials and integrals. This implies that the inductance is a *dynamic element*. What happened in the past influences the present behavior. As we shall see soon, inductance stores energy.

Capacitors & Inductors

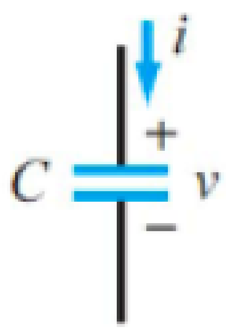
Remark: Resistors are *static elements*. A time varying voltage $v(t)$ across it generates a current $i(t)$ through it,

$$i(t) = \frac{v(t)}{R}.$$

The current at any time instant, say t_1, depends only on that time instant t_1 and nothing else.
The capacitors and inductors are *dynamic elements*. They involve integrals and differentials. As such, they have memory.

Capacitor

Passive element that stores energy in electric field



$$q = Cv$$

$$i = C\frac{dv}{dt}$$

$$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$$

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C is called CAPACITANCE and is measured in FARADS.
Farad is a very large unit. Often micro, nano, and
pico farads are used in practice.
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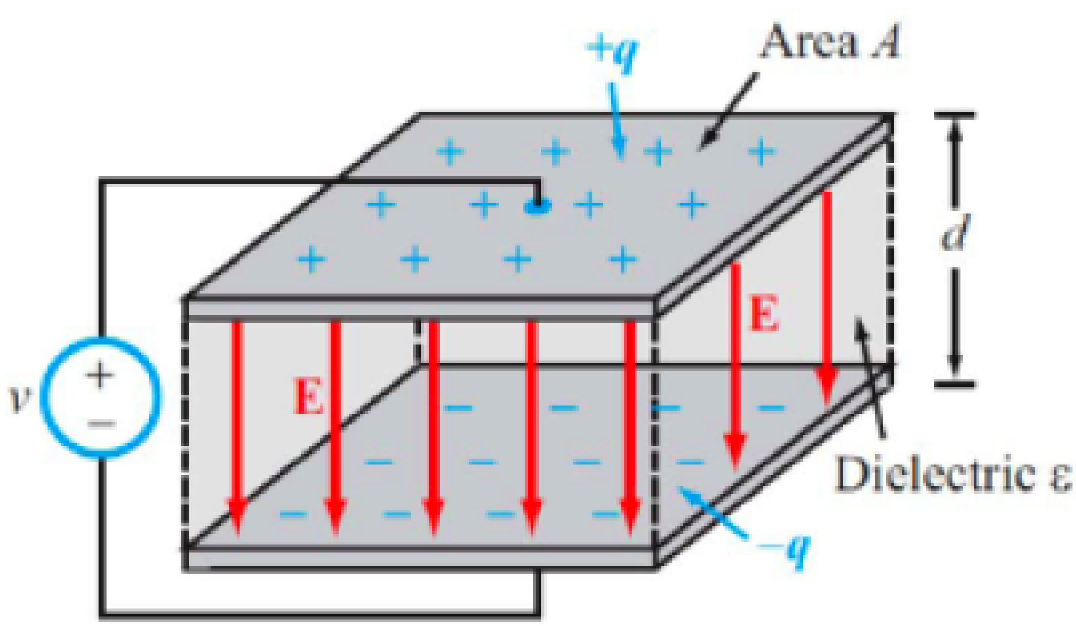
- Voltage on capacitor must be continuous (no abrupt change)

If current through the capacitor (capacitance) is impulsive, the voltage can change instantaneously. We will consider this next semester.

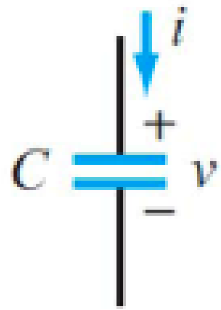
Often, the words 'capacitor' and 'capacitance' are used interchangeably. Capacitor is a physical element, it has a property known as *capacitance*. Note that resistor is a physical element, it has a property known as *resistance*. Similarly, inductor is a physical element, it has a property known as *inductance*.

Parallel plate capacitor

$$C = \frac{\varepsilon A}{d}$$



Energy Stored in Capacitor



$$i = C\frac{dv}{dt} \qquad q = Cv$$

$$p(t) = vi$$

$$= Cv\frac{dv}{dt} \qquad \text{(W)}.$$

Let the capacitor initially (at time t equals $-\infty$) be uncharged.

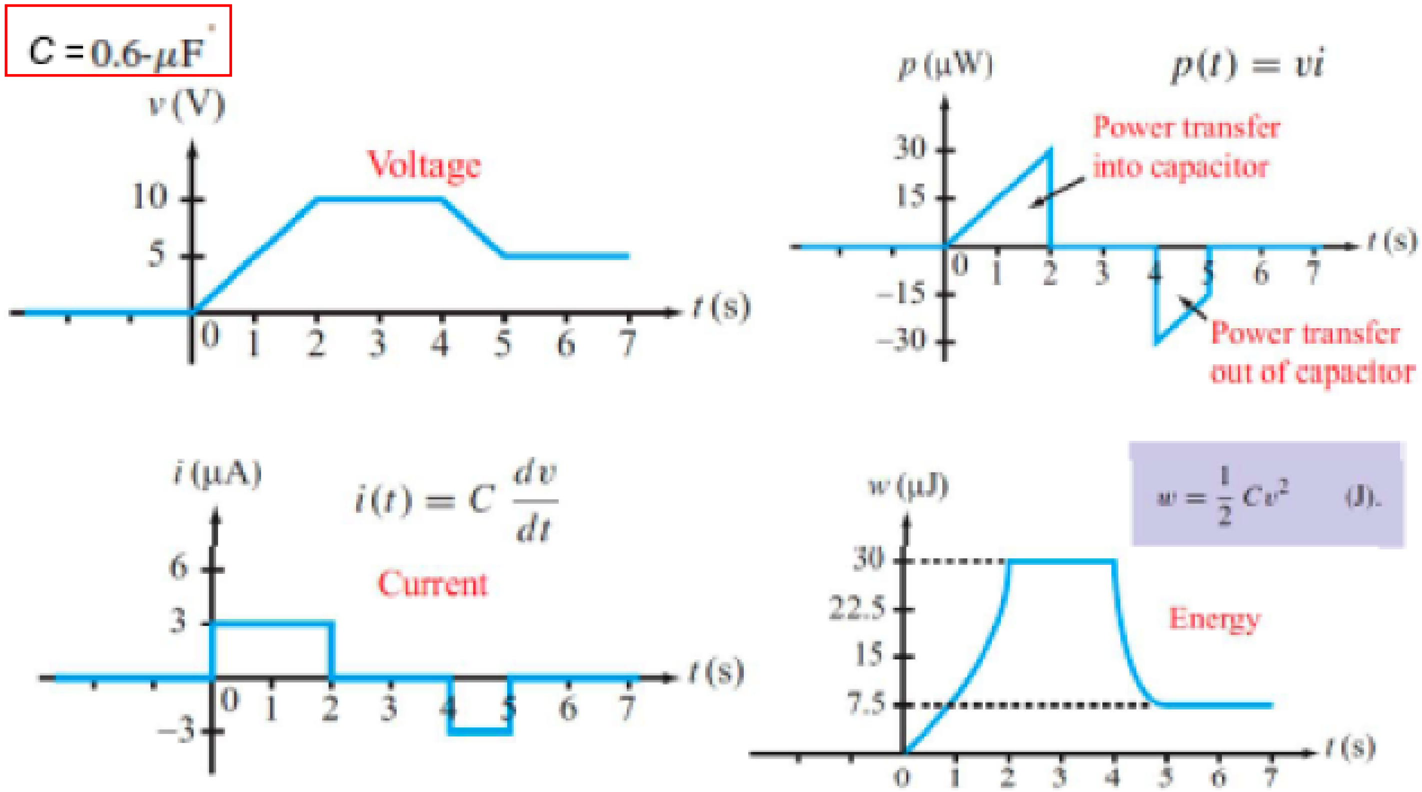
Then, work done W is given by

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t}\left(v\frac{dv}{dt}\right)dt$$

$$= C\int_{-\infty}^{t}\left[\frac{d}{dt}\left(\frac{1}{2}v^2\right)\right]dt,$$

which yields

$$w = \frac{1}{2}Cv^2 \qquad \text{(J)}.$$



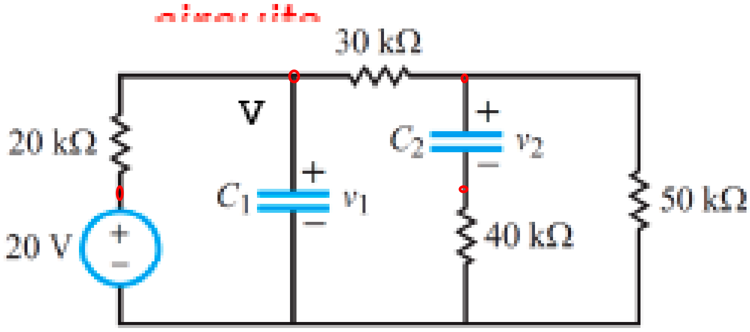
Explain why W is indeed energy stored.

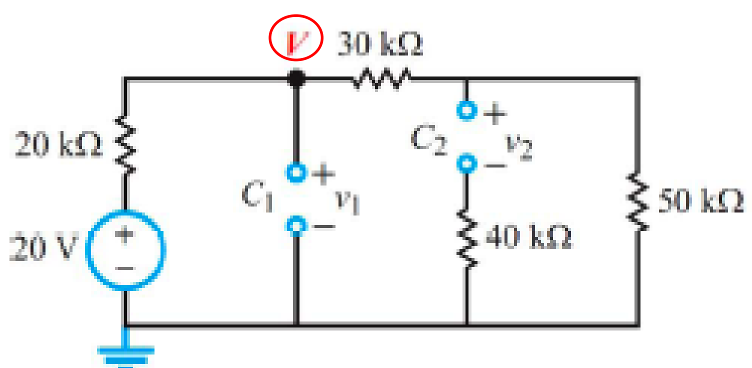
RC Circuits at dc

$$i = C\frac{dv}{dt}$$

In DC steady state, capacitance voltage is constant. This means $\frac{dv}{dt}=0$. Thus, the current i through the capacitance is zero. This means capacitance acts as an open circuit in DC steady state.

- At dc no currents flow through capacitors: open circuits





$$\frac{V-20}{20\times 10^3}+\frac{V}{(30+50)\times 10^3}=0,$$

which gives $V = 16$ V. Hence,

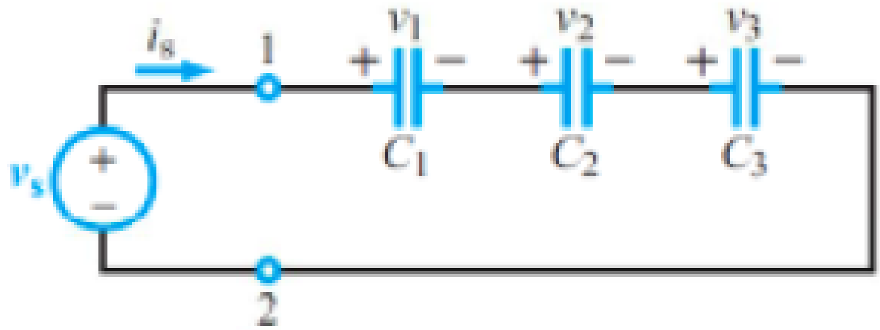
$$v_1 = V = 16\text{ V}.$$

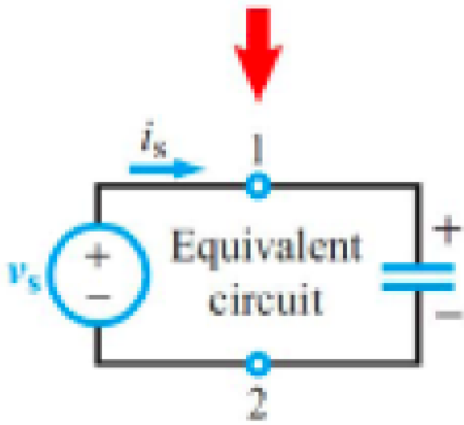
Through voltage division,

$$v_2 = \frac{V\times 50\text{k}}{(30+50)\text{k}} = \frac{16\times 50}{80} = 10\text{ V}.$$

Capacitors in Series

Combining In-Series Capacitors





$$v_s = v_1 + v_2 + v_3$$

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)^{-1}$$

current same through each capacitor

$$i_s = C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt} = C_3 \frac{dv_3}{dt}.$$

Also,

$$i_s = C_{eq} \frac{dv_s}{dt}$$

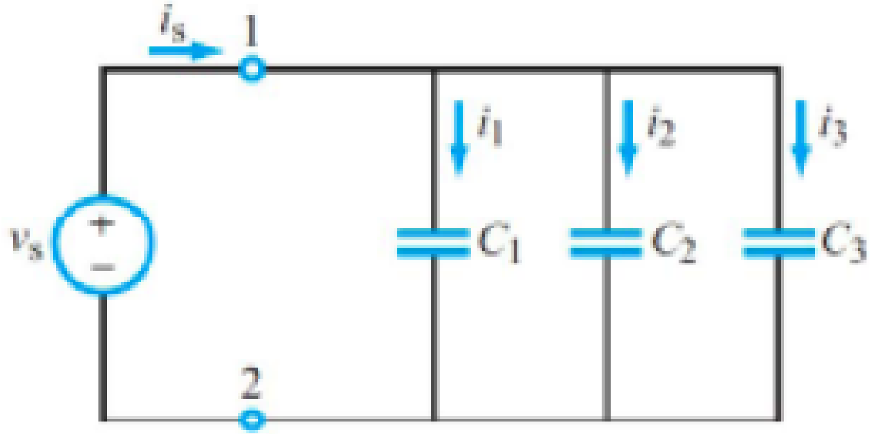
$$= C_{eq} \left(\frac{dv_1}{dt} + \frac{dv_2}{dt} + \frac{dv_3}{dt}\right)$$

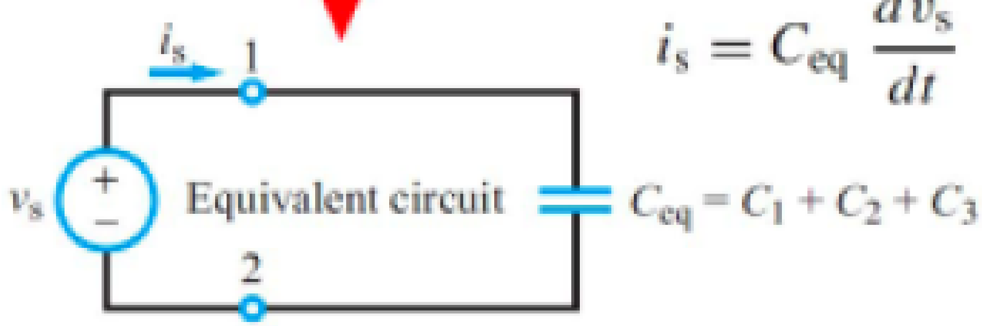
$$= C_{eq} \left(\frac{i_s}{C_1} + \frac{i_s}{C_2} + \frac{i_s}{C_3}\right),$$

Cacelling i_s on both sides, and rearranging, we get

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

Capacitors in Parallel





$$i_s = C_{eq} \frac{dv_s}{dt}$$

voltage same across each capacitor

$$i_s = i_1 + i_2 + i_3$$

$$= C_1 \frac{dv_s}{dt} + C_2 \frac{dv_s}{dt} + C_3 \frac{dv_s}{dt}$$

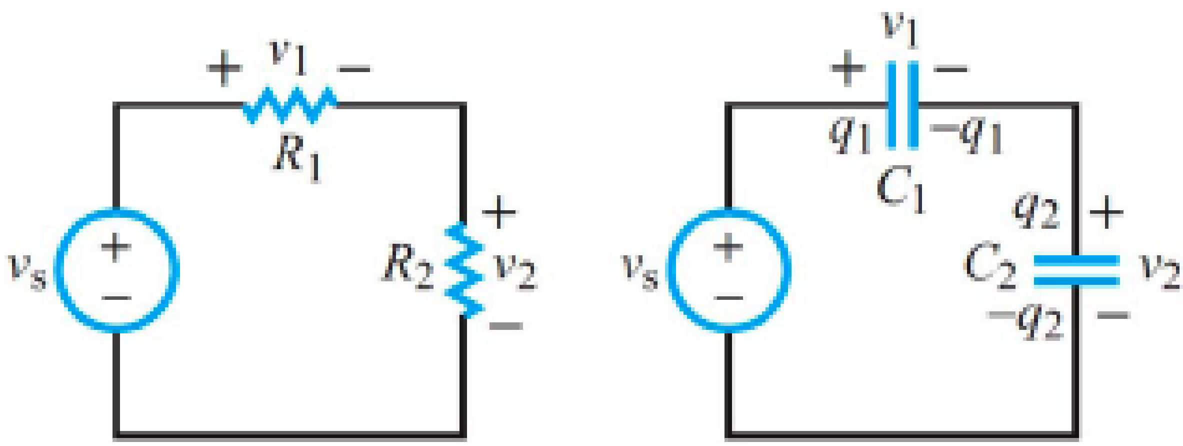
$$i_s = C_{eq} \frac{dv_s}{dt}$$

Comparing the above equations, we get $C_{eq} = C_1 + C_2 + C_3$

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N}$$

Voltage Division

Often, capacitors are used for voltage division rather than resistors. Capacitors are lossless.



(a) $v_1 = \left(\dfrac{R_1}{R_1 + R_2}\right) v_s$

$v_2 = \left(\dfrac{R_2}{R_1 + R_2}\right) v_s$

(b) $v_1 = \left(\dfrac{C_2}{C_1 + C_2}\right) v_s$

$v_2 = \left(\dfrac{C_1}{C_1 + C_2}\right) v_s$

Actually, q1=q2.
Both are integral of current through them.
Voltage division follows by using this concept.

$$v_1 = \frac{1}{C_1}\int i\,dt$$

$$v = \frac{1}{C_{eq}}\int i\,dt$$

$$\frac{v_1}{v} = \frac{C_{eq}}{C_1}$$

$$v_1 = \frac{C_{eq}}{C_1}v = \frac{C_2}{C_1 + C_2}v$$

Inductors

Passive element that stores energy in magnetic field

$$v = L\frac{di}{dt}$$

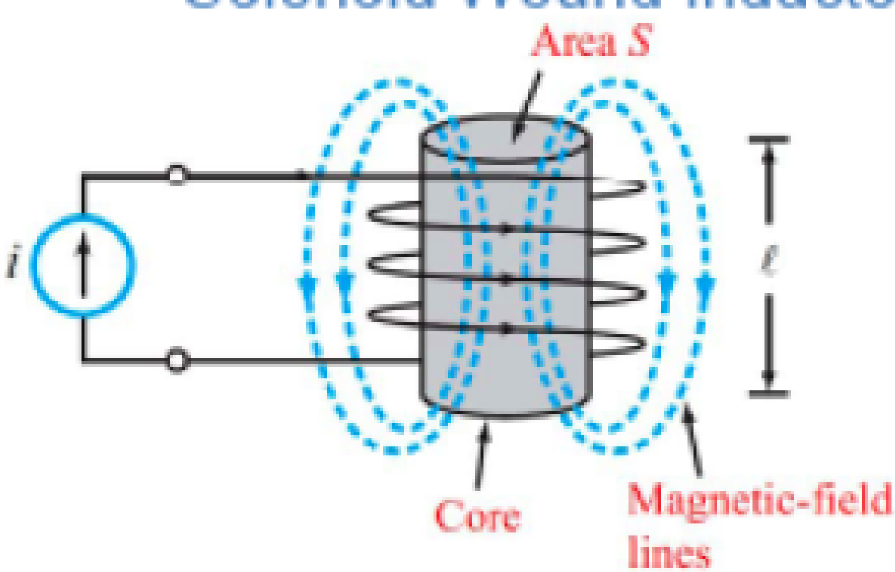
$$i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$

L is called INDUCTANCE and is measured in HENRIES.

- Current through inductor must be continuous (no abrupt change)

If voltage across L is impulsive, current can change suddenly.

Solenoid Wound Inductor



$$L = \frac{N^2 \mu A}{l}$$

Energy Stored in Inductor

$$v = L\frac{di}{dt}$$

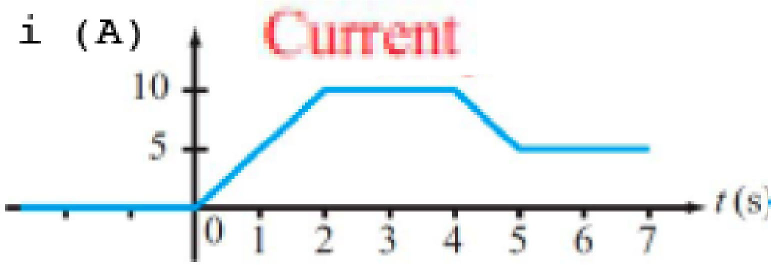
$$\text{Power} = p = vi = L\frac{di}{dt}\,i$$

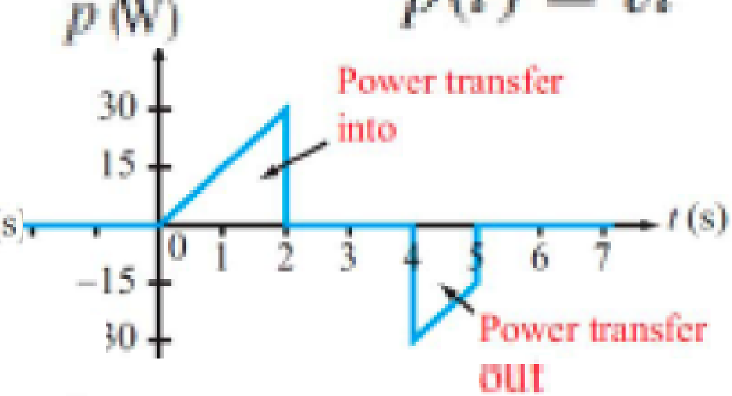
Assume that initial inductance current i at time $t = -\infty$ is zero. Then, work done W is given by

$$\begin{aligned} W &= \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} L\frac{di}{dt}\,i\,dt \\ &= L\int_{-\infty}^{t}\left[\frac{d}{dt}\left(\frac{1}{2}i^2\right)\right]dt \\ &= \frac{1}{2}Li^2. \end{aligned}$$

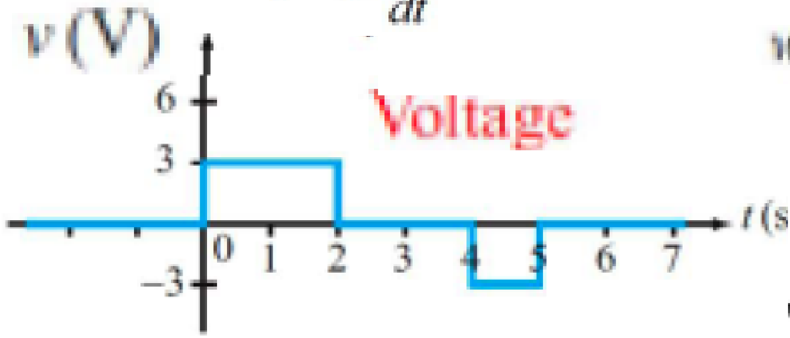
L=0.6 H

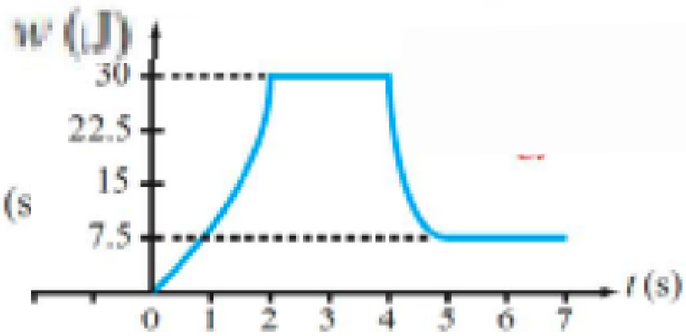
$$p(t) = vi$$





$$v = L\frac{di}{dt}$$



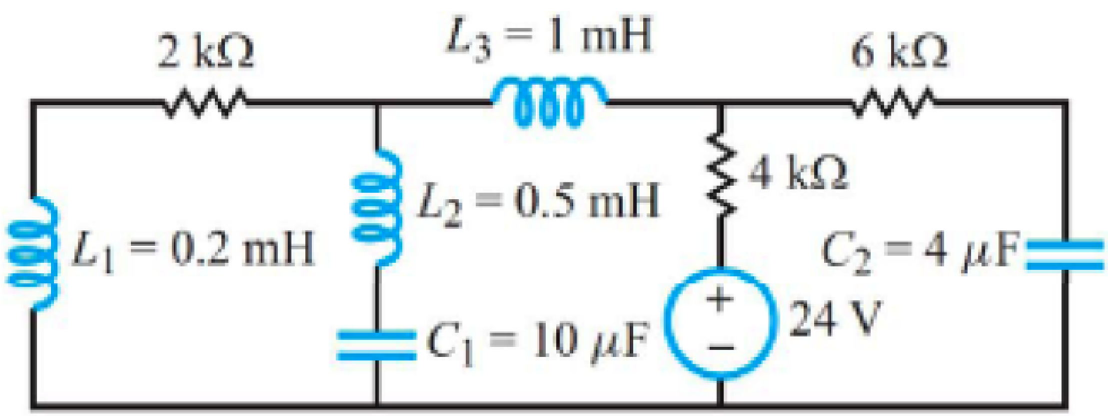


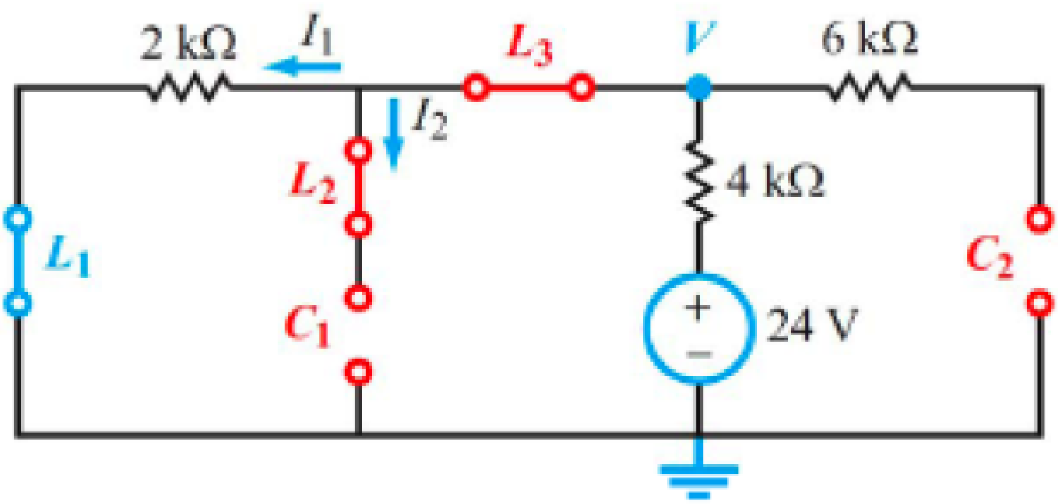
$$W = \frac{1}{2}Li^2$$

RLC Circuits at dc

$$v = L\frac{di}{dt}$$

In DC steady state, inductor current is constant. Hence $\frac{di}{dt} = 0$.
Thus, the voltage across inductance is zero.
Inductance in DC steady state acts as a SHORT CIRCUIT.





$$I_1 = \frac{24}{(2+4)\text{k}} = 4\text{ mA}$$

node voltage V is

$$V = 24 - (4 \times 10^{-3} \times 4 \times 10^{3}) = 8\text{ V}.$$

Hence, the amounts of energy stored in C_1, C_2, L_1, L_2, and are

$$C_1: \quad W = \frac{1}{2}C_1V^2 = \frac{1}{2} \times 10^{-5} \times 64 = 0.32 \text{ mJ},$$

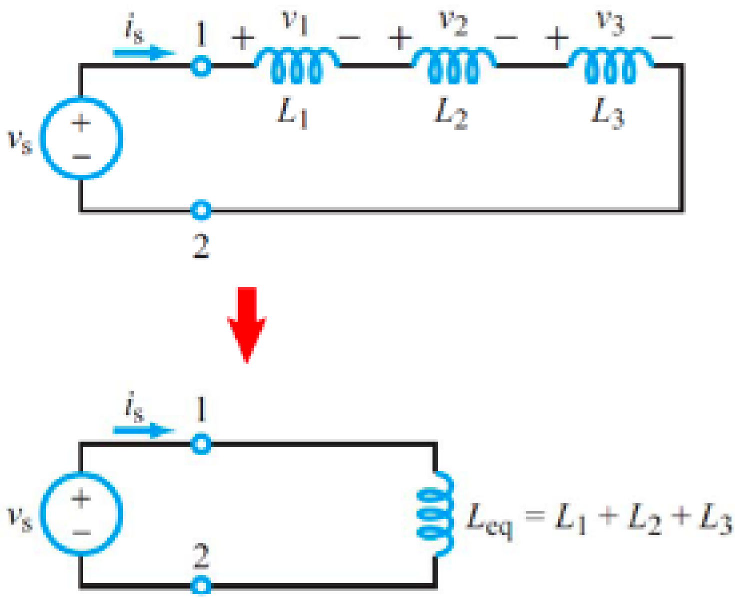
$$C_2: \quad W = \frac{1}{2}C_2V^2 = \frac{1}{2} \times 4 \times 10^{-6} \times 64 = 0.128 \text{ mJ},$$

$$L_1: \qquad W = \tfrac{1}{2}L_1I_1^2 = (0.1)10^{-3}4^210^{-6} = 1.6 \times 10^{-9} \text{ J}$$

$$L_2: \qquad W = \tfrac{1}{2}L_2I_2^2 = (0.25)10^{-3}0^2 = 0 \text{ J}$$

$$L_3: \qquad W = \tfrac{1}{2}L_3I_1^2 = (0.5)10^{-3}4^210^{-6} = 8 \times 10^{-9} \text{ J}$$

Inductors in Series



Use KVL, current is same through all inductors

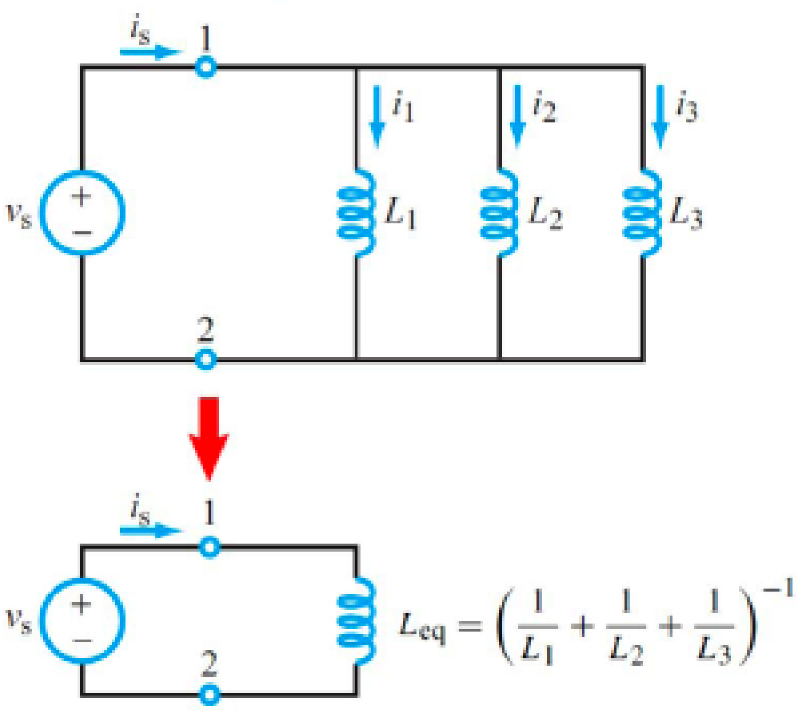
$$v_s = v_1 + v_2 + v_3$$
$$= L_1 \frac{di_s}{dt} + L_2 \frac{di_s}{dt} + L_3 \frac{di_s}{dt}$$
$$= (L_1 + L_2 + L_3) \frac{di_s}{dt},$$
$$v_s = L_{eq} \frac{di_s}{dt}$$

Comparing the above equations, we get

$$L_{eq} = L_1 + L_2 + L_3$$

Inductors in Parallel

Combining In-Parallel Inductors



Voltage is same across all inductors

Inductors add together in the same way resistors do

Show this as HW.

332:221 Principles of Electrical Engineering I

Basic properties of R, L, and C

Both voltage v and current i are considered as functions of time t

Property	**R**	**L**	**C**
v - i relationship	$v = Ri$	$v = L\frac{di}{dt}$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$
i - v relationship	$i = \frac{v}{R}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$	$i = C\frac{dv}{dt}$
p power consumed	$p = vi = i^2R$	$p = vi = Li\frac{di}{dt}$	$p = vi = Cv\frac{dv}{dt}$
w energy stored	$w = 0$	$w = \frac{1}{2}Li^2$	$w = \frac{1}{2}Cv^2$
Series connection	$R_{eq} = R_1 + R_2$	$L_{eq} = L_1 + L_2$	$C_{eq} = \frac{C_1C_2}{C_1+C_2}$
Parallel connection	$R_{eq} = \frac{R_1R_2}{R_1+R_2}$	$L_{eq} = \frac{L_1L_2}{L_1+L_2}$	$C_{eq} = C_1 + C_2$
DC Steady State Behavior	$v = Ri$, No change	$v = 0$ Short Circuit	$i = 0$ Open Circuit
Can v change instantaneously without i being an impulse?	Yes	Yes	No
Can i change instantaneously without v being an impulse?	Yes	No	Yes
AC Steady State behavior	**To be discussed**	**To be discussed**	**To be discussed**

Resistance is a static element and has no memory.
On the other hand, both capacitance and inductance are dynamic elements and have memory.

Principles of Electrical Engineering I

Integrating and Differentiating Circuits

Integrating Circuits

Passive Integrating Circuit:

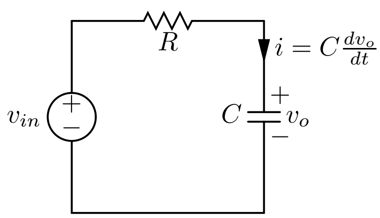


Figure 1

Consider the RC circuit of Figure 1.
Its equation is

$$v_{in} = RC\frac{dv_o}{dt} + v_o.$$

Suppose $RC >> 1$. This renders v_o small in comparison with v_{in}. Then, we have

$$v_{in} = RC\frac{dv_o}{dt}.$$

This implies that

$$v_o(t) = v_o(t_0) + \frac{1}{RC}\int_{t_0}^{t} v_{in}(\tau)d\tau.$$

Active Integrating Circuit:

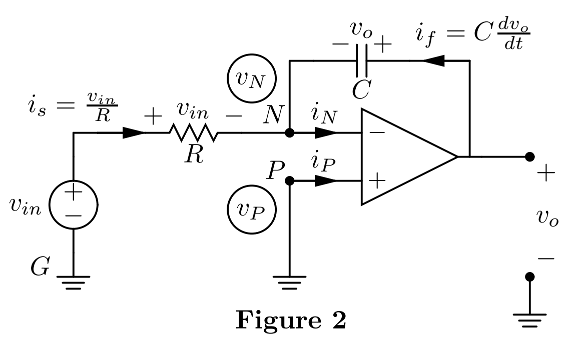


Figure 2

For an ideal Op-Amp, we know that $v_P - v_N = 0$ and $i_P = -i_N = 0$. Since $v_P = 0$, this imples that $v_N = 0$. Then, the circuit equation is

$$C\frac{dv_o}{dt} + \frac{v_{in}}{R} = 0.$$

This implies that

$$v_o(t) = -\frac{1}{RC}\int v_{in}(\tau)d\tau.$$

Differenciating Circuits

Passive Differenciating Circuit:

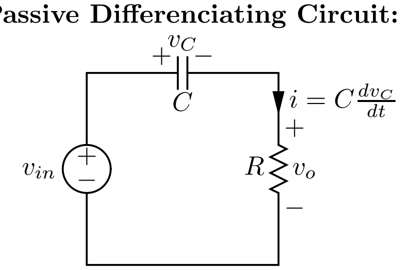


Figure 3

Consider the RC circuit of Figure 3.
Its equations are

$$C\frac{dv_C}{dt} = \frac{v_o}{R}, \qquad v_{in} = v_C + v_o.$$

By differentiating the second equation, and then substituting the first, we get

$$\frac{dv_{in}}{dt} = \frac{v_o}{RC} + \frac{dv_o}{dt}.$$

Suppose $\frac{1}{RC} >> 1$. This renders $\frac{dv_o}{dt}$ small and negligible in comparison with $\frac{v_o}{RC}$. Then, we have

$$v_o = RC\frac{dv_{in}}{dt}.$$

Active Differenciating Circuit:

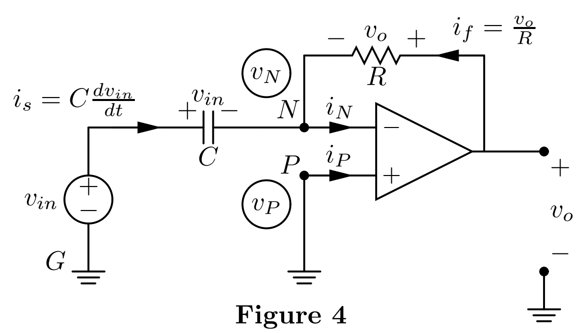


Figure 4

For an ideal Op-Amp, we know that $v_P - v_N = 0$ and $i_P = -i_N = 0$. Since $v_P = 0$, this imples that $v_N = 0$. Then, the circuit equation is

$$C\frac{dv_{in}}{dt} + \frac{v_o}{R} = 0.$$

This implies that

$$v_o(t) = -RC\frac{dv_{in}}{dt}.$$

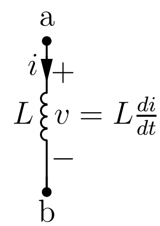
This is a Home Work Problem to be read and understood.

Principles of Electrical Engineering I

Current, Voltage, Power, and Energy associated with an Inductance

Example: Consider an inductance L in Henries. Let the current $i(t)$ through it be $I_m \sin(\omega t)$ where I_m is the amplitude and ω is the angular frequency in radians per second. Write down explicit expressions for the voltage $v(t)$ across it at time t, the power $p(t)$ consumed by it at time t, and the energy $e(t)$ consumed by it during the time interval from 0 to t.

The voltage $v(t)$ across it at time t is given by



Time Domain

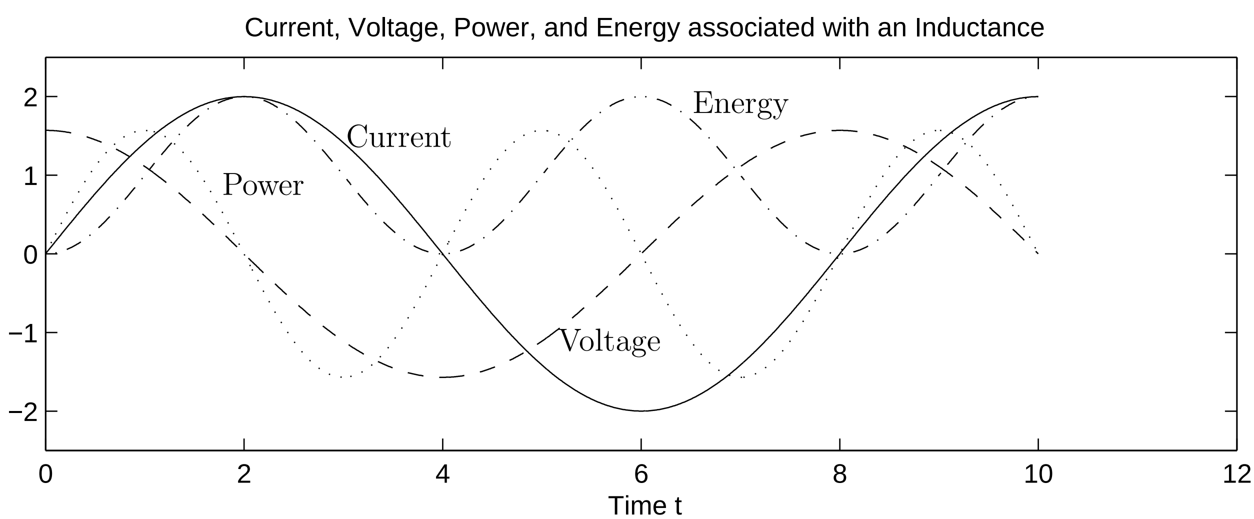
$$v(t) = L\frac{di}{dt} = \omega L I_m \cos(\omega t).$$

The power $p(t)$ consumed by it at time t is given by

$$p(t) = v(t)i(t) = \omega L I_m^2 \sin(\omega t)\cos(\omega t) = \frac{1}{2}\omega L I_m^2 \sin(2\omega t).$$

The energy $e(t)$ consumed by it during the time interval from 0 to t is given by

$$\begin{aligned} e(t) &= \int_0^t p(t)dt = \frac{1}{2}\omega L I_m^2 \int_0^t \sin(2\omega t)dt \\ &= \frac{1}{4}L I_m^2 \left[1 - \cos(2\omega t)\right] = \frac{1}{2}L I_m^2 \sin^2(\omega t) = \frac{1}{2}L i^2(t). \end{aligned}$$



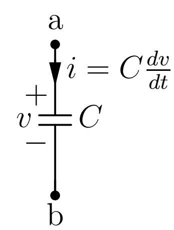
Thw above figure shows the graphs of
current $i(t) = I_m \sin(\omega t)$ (solid line graph),
voltage $v(t) = \omega L I_m \cos(\omega t)$ (dashed line graph),
the power $p(t) = \frac{1}{2}\omega L I_m^2 \sin(2\omega t)$ (dotted line graph), and
the energy $e(t) = \frac{1}{2}L i^2(t)$ (dash dotted line graph).
For numerical computations, we used $I_m = 2$ A, $\omega = 0.25\pi$ radians per second, and $L = 1$ Henry.

This is a Home Work Problem to be read and understood.

Principles of Electrical Engineering I

Voltage, Current, Power, and Energy associated with a Capacitance

Example: Consider a Capacitance C in Farads. Let the voltage $v(t)$ across it be $V_m \sin(\omega t)$ where V_m is the amplitude and ω is the angular frequency in radians per second. Write down explicit expressions for the current $i(t)$ through it at time t, the power $p(t)$ consumed by it at time t, and the energy $e(t)$ consumed by it during the time interval from 0 to t.



Time Domain

The current $i(t)$ through it at time t is given by

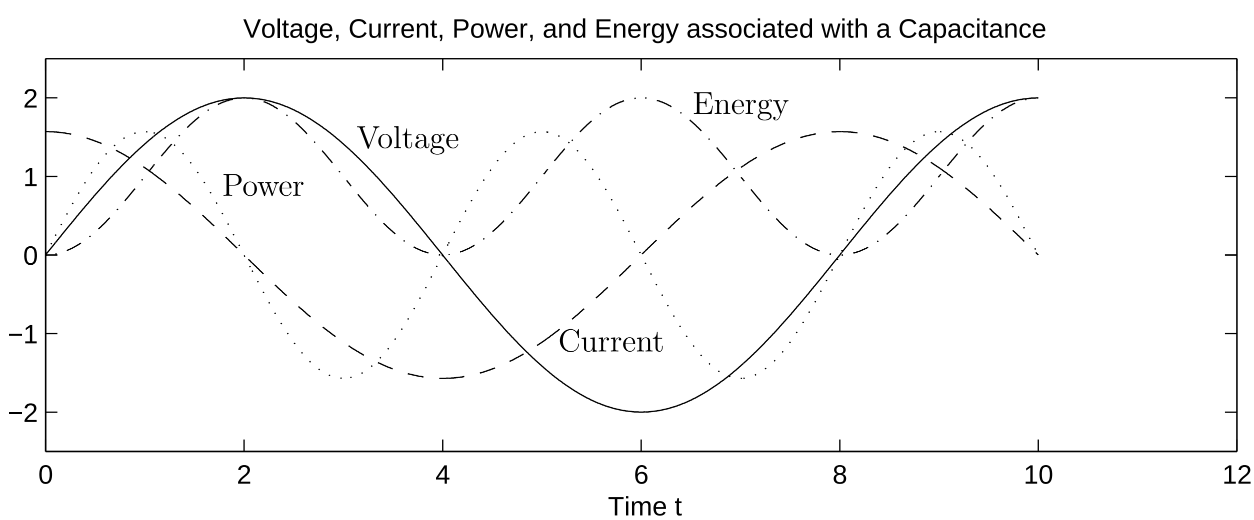
$$i(t) = C\frac{dv}{dt} = \omega C V_m \cos(\omega t).$$

The power $p(t)$ consumed by it at time t is given by

$$p(t) = v(t)i(t) = \omega C V_m^2 \sin(\omega t)\cos(\omega t) = \frac{1}{2}\omega C V_m^2 \sin(2\omega t).$$

The energy $e(t)$ consumed by it during the time interval from 0 to t is given by

$$\begin{aligned} e(t) &= \int_0^t p(t)dt = \frac{1}{2}\omega C V_m^2 \int_0^t \sin(2\omega t)dt \\ &= \frac{1}{4}C V_m^2 [1 - \cos(2\omega t)] = \frac{1}{2} C V_m^2 \sin^2(\omega t) = \frac{1}{2} C v^2(t). \end{aligned}$$



Thw above figure shows the graphs of
voltage $v(t) = V_m \sin(\omega t)$ (solid line graph),
current $i(t) = \omega C V_m \cos(\omega t)$ (dashed line graph),
the power $p(t) = \frac{1}{2}\omega C V_m^2 \sin(2\omega t)$ (dotted line graph), and
the energy $e(t) = \frac{1}{2} C v^2(t)$ (dash dotted line graph).
For numerical computations, we used $V_m = 2$ V, $\omega = 0.25\pi$ radians per second, and $C = 1$ Farad.

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Suggested HW Problems in the text book by Ulaby and Maharbiz

Partial answers are in Appendix E

These problems are not collected and graded.

Problem 5.2 Exercise to get familiarity with Step function

Problem 5.7 Time Constant of an exponential

Problem 5.15 RC Circuit in DC steady state

Problem 5.19 Capacitors Series-Parallel

Problem 5.25 Voltage signal across an inductance

Problem 5.29 RLC Circuit in DC steady state

This is a Home Work problem which is collected and graded.

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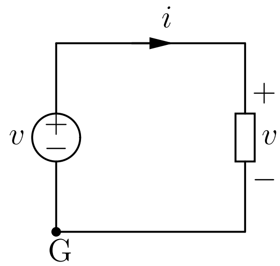
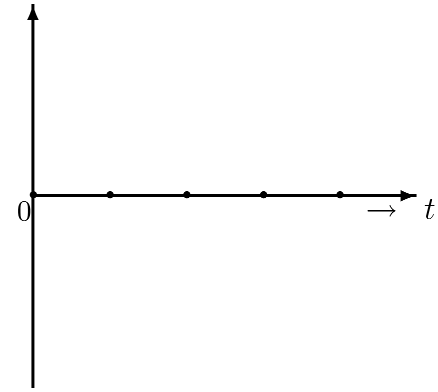
Voltage, Current, Power, Energy

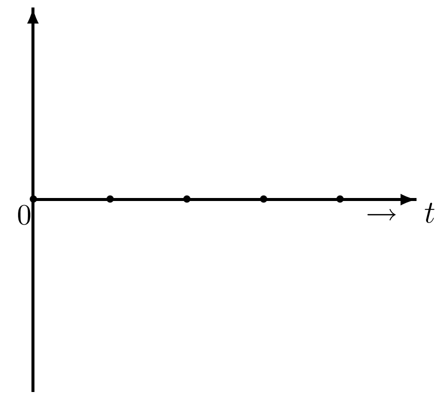


The voltage across an element is given by $v(t) = 100\sin(2\pi\,100t)$ while current through it is $i(t) = 2\cos(2\pi\,100t)$. Sketch the voltage $v(t)$, current $i(t)$, power consumed $p(t)$, and energy consumed $w(t)$ with respect to time. All sketches should start at time $t = 0$ and should show at least one cycle of the expected wave form. **Indicate the scales on both the axes of your graphs properly**.

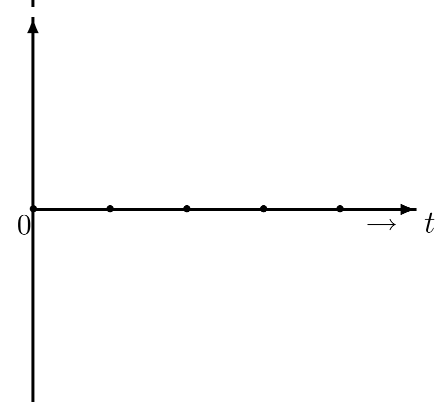
(a) The voltage v in Volts:



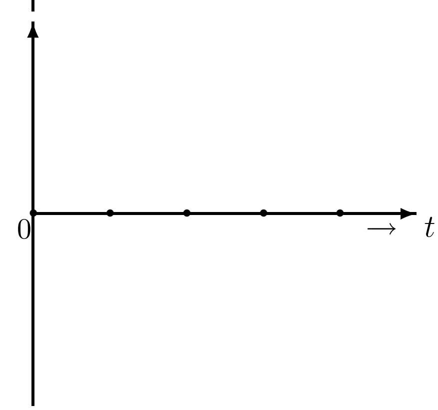
(b) The current i in Amperes:



(c) The power consumed p in Watts:

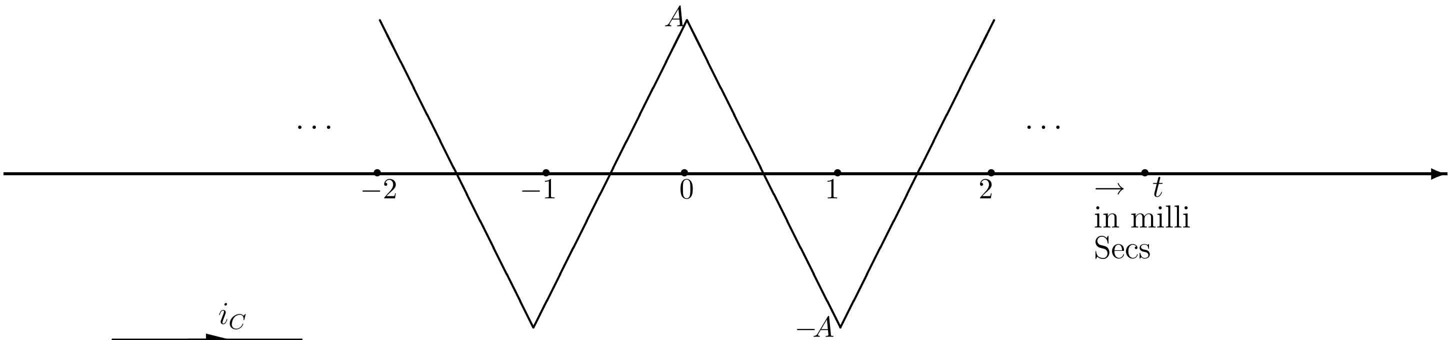


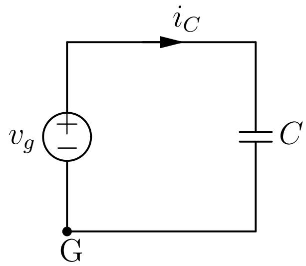
(d) The energy consumed W in Joules:



This is a Home Work problem which is collected and graded.

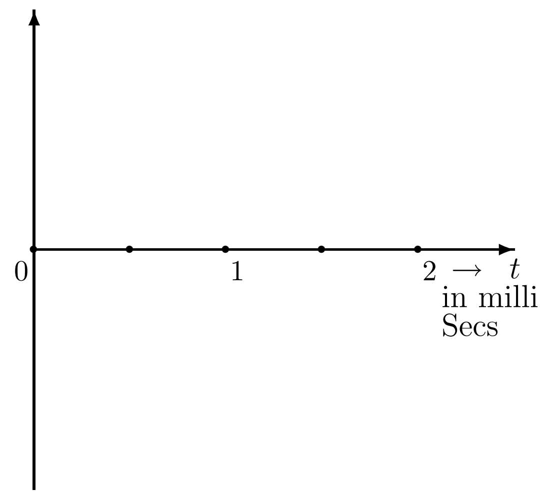
Symmetrical triangular waveform $v_g(t)$



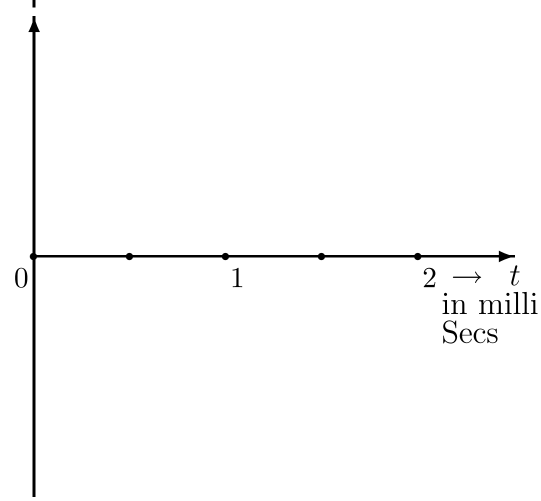


A voltage signal $v_g(t)$ having a symmetrical triangular wave form (with a period of 2 milli secs) is shown above. It is across a capacitor as shown on the left. Let $A = 100$V and $C = 1\mu$F (micro Farad). Draw to scale (indicate the scale on the vertical axis) the following items for the time period shown:

(a) The current i_C in Amperes:



(b) The power p consumed by the capacitor in watts:



(c) The energy stored in the capacitor at $t = 0$:

(d) The energy stored in the capacitor at $t = 0.5$ milli Secs:

(e) The energy stored in the capacitor at $t = 1$ milli Secs:

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This is a Home Work problem which is collected and graded.
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Determine the equivalent capacitance between the terminals a and b of Figure A.

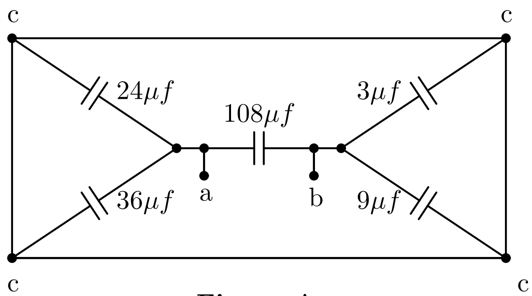

Figure A

Determine the equivalent inductance between the terminals a and b of Figure B.

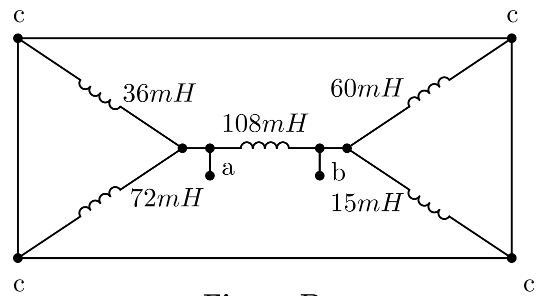

Figure B