1. **Random Phase Signals**: Consider the continuous time process \( X(t) = A \cos(\omega_0 t + \phi) \), where \( \omega_0 \) is a known frequency, \( \phi \) is uniformly distributed in \((0, 2\pi)\) and independently of the random variable \( A \).

(a) (10 points) Is \( X(t) \) is a wide sense stationary (WSS) process? Hint: You may want to use the fact that \( \cos(A) \cos(B) = (\cos(A-B) + \cos(A+B))/2 \).

(b) (10 points) If we take \( \phi \) to be a known constant, is \( X(t) \) WSS?

2. **Stationary Autoregressive-1 Process**: Let \( e(t) \), for \( t = 0, \pm 1, \pm 2, \cdots \) be a real-valued white noise process (i.e. a sequence of uncorrelated real-valued random variables with zero mean and variance \( \sigma_e^2 \)). We may define a process \( X(t) \) by the stochastic difference equation:

\[
X(t) = \alpha X(t-1) + e(t) \quad \text{for} \quad t = 0, \pm 1, \pm 2, \cdots
\]

where \( \alpha \) is real with \(|\alpha| < 1\). Observe that \( X(t) \) is simply the output of an IIR filter with input \( e(t) \), and hence \( X(t) \) is wide-sense stationary (WSS), since linear filtering of WSS preserves WSS. (You do not have to prove this).

(a) (10 points) Show that \( X(t) \) satisfies

\[
X(t) = e(t) + \alpha e(t-1) + \alpha^2 e(t-2) + \cdots + \alpha^{m-1} e(t-m+1) + \alpha^m X(t-m).
\]

(b) (10 points) Show that \( X(t) \) has an infinite series expansion

\[
X(t) = \sum_{j=0}^{\infty} \alpha^j e(t-j)
\]

by showing that

\[
E \left[ \left( X(t) - \left( e(t) + \alpha e(t-1) + \alpha^2 e(t-2) + \cdots + \alpha^{m-1} e(t-m+1) \right) \right)^2 \right] \to 0
\]

as \( m \to \infty \). Note: Explicitly explain why there is convergence.

(c) (10 points) Show that the autocorrelation function \( R_X(k) \) has the properties

i. \( R_X(0) = E[X(t)^2] = \frac{\sigma_e^2}{1-\alpha^2} \).

ii. For \( k = 1, 2, \cdots \) show that \( R_X(k) = \alpha R_X(k-1) \), and, for \( k = \pm 1, \pm 2, \cdots \), that

\[
R_X(k) = \frac{\sigma_e^2 \alpha^{|k|}}{1-\alpha^2}.
\]

(d) (10 points) Finally, show that the power spectral density \( S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-ik\omega} \) has the representation

\[
S_X(\omega) = \frac{\sigma_e^2}{1-2\alpha \cos(\omega) + \alpha^2}.
\]
(e) **(10 points)** Give a rough sketch of what $R_X(k)$ and $S_X(\omega)$ would look like for $\alpha = 0.7$ and $\alpha = -0.7$.

3. **Fun with the Correlation Matrix:** Let $X(k)$ be a complex zero mean, wide sense stationary process, where $k = 0, \pm 1, \pm 2, \cdots$. Define $x_M = [X(0), X(1), \cdots, X(M - 1)]^T$. We may define the $M$ dimensional correlation matrix by

$$R_M = E[x_M x_M^H].$$

(a) **(10 points)** Define $\hat{x}_M = [X(M - 1), X(M - 2), \cdots, X(1)]^T$. Show that the correlation matrix can be represented as

$$R_M = \begin{bmatrix} R_{M-1} & \mathbf{r} \\ \mathbf{r}^H & R(0) \end{bmatrix}$$

where $\mathbf{r} = E[x(0)\hat{x}_M^*]$.

(b) Next show that the eigenvalues of the correlation matrix $R_M$ are upper and lower bounded by the maximum and minimum values of the power spectrum. Let us denote the eigenvalues and eigenvectors of $R_M$ by $\lambda_j$ and $\mathbf{v}_j$ respectively, hence $R_M \mathbf{v}_j = \lambda_j \mathbf{v}_j$.

i. **(5 points)** Show that

$$\lambda_j = \frac{\mathbf{v}_j^H \mathbf{R}_M \mathbf{v}_j}{\mathbf{v}_j^H \mathbf{v}_j}.$$

ii. **(5 points)** Let us denote the coefficients of $\mathbf{v}_j$ by $v_j(k)$, i.e.

$$\mathbf{v}_j = [v_j(0), v_j(1), \cdots, v_j(M - 1)]^T.$$ 

Show that

$$\mathbf{v}_j^H \mathbf{R}_M \mathbf{v}_j = \sum_{k=0}^{M-1} \sum_{l=0}^{M-1} v_j^*(k) R_X(k - l) q_j(l).$$

iii. **(5 points)** Show

$$\mathbf{v}_j^H \mathbf{R}_M \mathbf{v}_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega) |V_j(\omega)|^2 d\omega,$$

where

$$V_j(\omega) = \sum_{k=0}^{M-1} v_j(k) e^{-ik\omega}.$$ 

Similarly, argue that

$$\mathbf{v}_j^H \mathbf{v}_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} |V_j(\omega)|^2 d\omega.$$

iv. **(5 points)** Now argue that

$$\min_{\omega} S_X(\omega) \leq \lambda_j \leq \max_{\omega} S_X(\omega).$$