Supplemental Problem Solutions:

1. Start by calculating \( R_X(k) \):

\[
E[X(n + k)X(n)] = E \left[ \sum_{i=0}^{p} \alpha_i W(n - i) \sum_{j=0}^{p} \alpha_j W(n + k - j) \right] \\
= \sum_i \sum_j \alpha_i \alpha_j E[W(n - i)W(n + k - j)] \\
= \sum_i \sum_j \alpha_i \alpha_j R_W(k - j + i)
\]

Observe that for \( k > p \) we have \( k - j + i > 0 \) and hence \( R_W(k - j + i) = 0 \) (since \( W \) is white). Similarly for \( k < -p \).

(b) For \(|k| \leq p\) we have

\[
R_X(k) = \sigma^2 \sum_{i=0}^{p} \sum_{l=0}^{p} \alpha_i \alpha_l R_W(k - l + i)
\]

\[
= \sigma^2 \sum_{i=0}^{p} \sum_{l=0}^{p} \alpha_i \alpha_l \delta(k - l + i)
\]

\[
= \sigma^2 \sum_{l=0}^{p} \alpha_l \delta(k - l)
\]

\[
= \sigma^2 \sum_{l=-\infty}^{\infty} \alpha_l \delta(l - k) \text{ where } \alpha_l = 0 \text{ when } l < 0 \text{ and } l > p
\]

Now \( S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-ik\omega} \). Expanding we get

\[
S_X(\omega) = \sigma^2 \sum_k \sum_l \alpha_l \alpha_{l-k} e^{-i\omega l}
\]

Set \( m = l - k \) to get

\[
= \sigma^2 \sum_l \alpha_l e^{-i\omega l} \sum_m \alpha_m e^{i\omega m} = H(\omega) H^*(\omega) \sigma^2.
\]

(c) The transfer function is \( H(\omega) = \sum_k \alpha_k e^{-i\omega} \). From class \( S_X(\omega) = |H(\omega)|^2 S_W(\omega) = \sigma^2 |H(\omega)|^2 \).

2. As was correctly pointed out in class, this problem is only valid if \( R \) has distinct eigenvalues. Typically, you will have \( M \) distinct eigenvalues and corresponding to these \( M \) distinct eigenvalues are \( M \) orthogonal eigenvectors. We proceed with showing that these \( M \) eigenvectors are orthogonal.

Start with \( Rv_i = \lambda_i v_i \) for \( i = 1, \cdots, M \). For \( i \neq j \) we must show \( v_j^H v_i = 0 \). Premultiply \( Rv_i = \lambda_i v_i \) by \( v_j^H \) to get

\[
v_j^H Rv_i = \lambda_i v_j^H v_i.
\]

Since the \( R \) is Hermitian, we have \( R^H = R^* \), and the eigenvalues are real. Thus we may get

\[
v_j^H R = \lambda_j v_j^H.
\]

Postmultiplying this by \( v_i \) we get

\[
v_j^H Rv_i = \lambda v_j^H v_i.
\]
Subtracting this from $v_j^H R v_i = \lambda_i v_j^H v_i$ gives

$$(\lambda_i - \lambda_j) v_j^H v_i = 0.$$  

Since the eigenvalues are distinct, this means that $v_j^H v_i = 0$.

3. Define $q = 1 - p$ for simplicity in notation. Start by observing

$$p_0 = (1 - p)^n = \left(1 - \frac{\alpha}{n}\right)^n \to e^{-\alpha}.$$  

Next, let us look at ratios $p_{k+1}/p_k$.

$$\frac{p_{k+1}}{p_k} = \frac{\binom{n}{k+1} p^{k+1} q^{n-k+1}}{\binom{n}{k} p^k q^{n-k}} = \frac{(n - k)p}{(k + 1)q} = \frac{(1 - k/n)\alpha}{(k + 1)(1 - \alpha/n)} \to \frac{\alpha}{k + 1}.$$  

Thus, the limiting probabilities satisfy

$$p_{k+1} = \frac{\alpha}{k + 1} p_k.$$  

Substituting $p_0 = e^{-\alpha}$ and iterating will give the result, e.g.

$$p_1 = \alpha e^{-\alpha}$$

$$p_2 = \frac{\alpha}{2} p_1 = \frac{\alpha^2}{2} e^{-\alpha}.$$  

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