Problem 1.4.1

From the table we look to add all the disjoint events that contain $H_0$ to express the probability that a caller makes no hand-offs as

\[ P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5 \]

In a similar fashion we can express the probability that a call is brief by

\[ P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6 \]

The probability that a call is long or makes at least two hand-offs is

\[ P[L \cup H_2] = P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] = 0.1 + 0.1 + 0.2 + 0.1 = 0.5 \]

Problem 1.4.4

consequence of part 4 of Theorem 1.4.

(a) Since $A \subset A \cup B$, $P[A] \leq P[A \cup B]$.

(b) Since $B \subset A \cup B$, $P[B] \leq P[A \cup B]$.

(c) Since $A \cap B \subset A$, $P[A \cap B] \leq P[A]$.

(d) Since $A \cap B \subset B$, $P[A \cap B] \leq P[B]$.

Problem 1.4.5

(a) For convenience, let $p_i = P[FH_i]$ and $q_i = P[VH_i]$. Using this shorthand, the six unknowns $p_0, p_1, p_2, q_0, q_1, q_2$ fill the table as

<table>
<thead>
<tr>
<th></th>
<th>$H_0$</th>
<th>$H_1$</th>
<th>$H_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$p_0$</td>
<td>$p_1$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>$V$</td>
<td>$q_0$</td>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
</tbody>
</table>

However, we are given a number of facts:

\[
\begin{align*}
    p_0 + q_0 &= 1/3 \\
    p_2 + q_2 &= 1/3 \\
    p_1 + q_1 &= 1/3 \\
    p_0 + p_1 + p_2 &= 5/12
\end{align*}
\]

Other facts, such as $q_0 + q_1 + q_2 = 7/12$, can be derived from these facts. Thus, we have four equations and six unknowns, choosing $p_0$ and $p_1$ will specify the other unknowns. Unfortunately, arbitrary choices for either $p_0$ or $p_1$ will lead to negative values for the other probabilities. In terms of $p_0$ and $p_1$, the other unknowns are

\[
\begin{align*}
    q_0 &= 1/3 - p_0 \\
    q_1 &= 1/3 - p_1 \\
    p_2 &= 5/12 - (p_0 + p_1) \\
    q_2 &= p_0 + p_1 - 1/12
\end{align*}
\]
Because the probabilities must be nonnegative, we see that
\[
0 \leq p_0 \leq 1/3 \\
0 \leq p_1 \leq 1/3 \\
1/12 \leq p_0 + p_1 \leq 5/12
\]

Although there are an infinite number of solutions, three possible solutions are:

\[
\begin{align*}
p_0 &= 1/3 & p_1 &= 1/12 & p_2 &= 0 \\
q_0 &= 0 & q_1 &= 1/4 & q_2 &= 1/3
\end{align*}
\]

and

\[
\begin{align*}
p_0 &= 1/4 & p_1 &= 1/12 & p_2 &= 1/12 \\
q_0 &= 1/12 & q_1 &= 3/12 & q_2 &= 3/12
\end{align*}
\]

and

\[
\begin{align*}
p_0 &= 0 & p_1 &= 1/12 & p_2 &= 1/3 \\
q_0 &= 1/3 & q_1 &= 3/12 & q_2 &= 0
\end{align*}
\]

(b) In terms of the \(p_i, q_i\) notation, the new facts are \(p_0 = 1/4\) and \(q_1 = 1/6\). These extra facts uniquely specify the probabilities. In this case,

\[
\begin{align*}
p_0 &= 1/4 & p_1 &= 1/6 & p_2 &= 0 \\
q_0 &= 1/12 & q_1 &= 1/6 & q_2 &= 1/3
\end{align*}
\]

Problem 1.6.7

(a) For any events \(A\) and \(B\), we can write the law of total probability in the form of

\[
P[A] = P[AB] + P[AB^c]
\]

Since \(A\) and \(B\) are independent, \(P[AB] = P[A]P[B]\). This implies

\[
\]

Thus \(A\) and \(B^c\) are independent.

(b) Proving that \(A^c\) and \(B\) are independent is not really necessary. Since \(A\) and \(B\) are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of \(A\) and \(B\) proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels \(A\) and \(B\) reversed.

(c) To prove that \(A^c\) and \(B^c\) are independent, we apply the result of part (a) to the sets \(A\) and \(B^c\). Since we know from part (a) that \(A\) and \(B^c\) are independent, part (b) says that \(A^c\) and \(B^c\) are independent.
Problem 1.7.10
caught. The tree resembles

\[
\begin{array}{c}
\text{p} \quad C_1 \\
1-p \quad C_1 \\
\text{p} \quad C_2 \\
1-p \quad C_2 \\
\text{p} \quad C_3 \\
1-p \quad C_3 \\
\vdots
\end{array}
\]

The various probabilities are
(a) \( P[C_1] = p \)
(b) \( P[C_2] = (1 - p)p \)
(c) A fish is caught on the \( n \)th cast if no fish were caught on the previous \( n - 1 \) casts. Thus,
\[
P[C_n] = (1 - p)^{n-1} p
\]

Problem 1.9.5
From the problem statement, we can conclude that the device components are configured in the following way.

\[
\begin{array}{c}
W_1 \quad W_2 \quad W_3 \\
\quad W_4 \quad W_5 \\
\quad \quad W_6
\end{array}
\]

To find the probability that the device works, we replace series devices 1, 2, and 3, and parallel devices 5 and 6 each with a single device labeled with the probability that it works. In particular,
\[
P[W_1W_2W_3] = (1 - q)^3
\]
\[
P[W_5 \cup W_6] = 1 - P[W_5^cW_6^c] = 1 - q^2
\]
This yields a composite device of the form

\[
\begin{array}{c}
(1-q)^3 \\
1-q \\
1-q^2
\end{array}
\]

The probability \( P[W'] \) that the two devices in parallel work is 1 minus the probability that neither works:
\[
P[W'] = 1 - q(1 - (1 - q)^3)
\]

Finally, for the device to work, both composite device in series must work. Thus, the probability the device works is
\[
P[W] = (1 - q(1 - q)^3)(1 - q^2)
\]