Instructions: The exam is a take home exam, and consists of two sections, each with three problems. Make sure to clearly state what problem you are solving at the top of each page, and turn in the solutions in sequential order (section 1, problem 1, then 2, then 3, then section 2, problem 1, problem 2, problem 3). The final exam is due by December 19th, at 3pm, and should be turned into Core 523 by that time.

You may use resources, such as the book, other books, the library and the Internet. However, you may not use each other, or other graduate students, or other professors.

Note: The exam will be graded out of 100 points, and will be scaled later to contribute a total of 30 points towards your final grade.

I. Conceptual Stochastics:

The following questions are each to be answered using a short paragraph. You should make an effort to explain as clearly as possible, while minimizing the amount of mathematics used in your explanation. Do not simply copy definitions out of the book, you should put the answers into your own words.

1. (5 points) Explain what stationarity means for a stochastic process.

2. (10 points) For a sequence of random variables, explain the difference between convergence in probability and almost sure convergence.

3. (10 points) Explain why Gaussian random variables occur frequently in nature.

II. Technical Questions:

1. Probability and Numbers: (25 points) Let us define \( Z \) to be the non-negative integers. Let us define by \( Z_N = \{1, 2, \cdots, N\} \subset Z \) to be the subset consisting of non-negative integers up through \( N \). Suppose two numbers \( a \) and \( b \) are randomly chosen from \( Z_N \), according to a uniform distribution over \( Z_N \). Calculate the probability that these two numbers are relatively prime (meaning, they do not share any of the same factors, i.e. \( \gcd(a, b) = 1 \)), and call this probability \( \zeta_N^{(2)} \). What is the limit of \( \zeta_N^{(2)} \) as \( N \to \infty \)? (Note: A complete, rigorous proof will require suitable stochastic convergence arguments).

2. Yet More Correlation Matrices: Suppose \( X(k) \) is a discrete time, wide sense stationary process with zero mean and covariance function \( r_X(k) \). Let \( R_M \) denote the \( M \times M \) correlation matrix for \( X(k) \). In practice, we cannot use \( R_M \) directly, but instead must rely on an approximation to \( R_M \). We now describe one common approach to estimating \( R_M \). Suppose \( x(k) \) is a data vector corresponding to time \( k \), given by \( x(k) = [x(k), x(k-1), \cdots, x(k-M+1)]^T \), where \( x(t) \) is a measured sample of the process \( X \) at time \( t \). Then \( R_M \) is estimated by

\[
\hat{R}_M = \frac{1}{N^2} \sum_{k=1}^{N} x^H(k)x(k),
\]

where \( N \) is sufficiently large to ensure an accurate estimate of the expected values. Suppose that we have an initial estimate \( \hat{R}_M(k-1) \) for the correlation matrix at time \( k-1 \). In order to adapt to slowly changing (and thus non-stationary) scenarios, in practice we update the estimated correlation matrix for time \( k \) by

\[
\hat{R}_M(k) = \alpha \hat{R}_M(k-1) + (1-\alpha)x^H(k)x(k),
\]

where \( 0 \leq \alpha \leq 1 \) is a weight known as the exponential forgetting factor.

(a) (5 points) Show that \( \hat{R}_M(k) \) can be represented as a perturbation of \( \hat{R}_M(k-1) \), i.e.

\[
\hat{R}_M(k) = \hat{R}_M(k-1) + \epsilon \mathbf{E}(k)
\]

for \( \epsilon = 1-\alpha \) and suitable \( \mathbf{E}(k) \).
Now define $v_j(k-1)$ and $\lambda_j(k-1)$ to be the $j$th eigenvector and eigenvalue of $\hat{R}_M(k-1)$. Throughout the rest of this problem, we assume that $v_j(k-1)$ have been normalized so that $\|v_j(k-1)\| = 1$, where $\|v\|$ is the Euclidean norm of $v$. According to matrix perturbation theory, the eigenvalues and eigenvectors of $\hat{R}_M(k)$ can be represented as a Taylor series expansion in terms of $\epsilon$. If we carry only first-order terms, then we have:

\[
\begin{align*}
    v_j(k) &= v_j(k-1) + \epsilon v_j^{(1)}(k-1) \\
    \lambda_j(k) &= \lambda_j(k-1) + \epsilon \lambda_j^{(1)}(k-1)
\end{align*}
\]

Show that

\[
\hat{R}_M(k-1)v_j^{(1)}(k-1) + \mathbf{E}(k)v_j(k-1) = \lambda_j(k-1)v_j^{(1)}(k-1) + \lambda_j^{(1)}(k-1)v_j(k-1).
\]

(c) (10 points) Show that

\[
\lambda_j^{(1)}(k-1) = v_j^H(k-1)\mathbf{E}(k)v_j(k-1),
\]

and hence we may update the eigenvalues of the matrix $\hat{R}_M(k)$ in terms of the eigenvalues and eigenvectors of $\hat{R}_M(k-1)$. To arrive at this, you may assume the eigenvalues of $\hat{R}_M(k-1)$ and $\hat{R}_M(k)$ are real. (An additional note, which is not part of the exam: the eigenvectors of the updated matrix may also be represented in terms of the eigenvectors and eigenvalues of the previous time matrix. This expansion is more involved to show.)

3. Zero Crossings: Let $Z_1, Z_2, \cdots, Z_N$ be a zero-mean strictly stationary real-valued stochastic process. The zero-crossing count in discrete time is defined as the number of symbol changes in the corresponding clipped binary time series $X_t$

\[
X_t = \begin{cases} 
    1 & \text{if } Z_t \geq 0 \\
    0 & \text{if } Z_t < 0
\end{cases}
\]

Define the number of zero crossings as

\[
D = \sum_{t=2}^{N}(X_t - X_{t-1})^2,
\]

and the zero crossing rate $r = D/(N-1)$. Define $p = P(X_t = 1)$ and $p_{11} = P(X_t = 1|X_{t-1} = 1)$.

(a) (5 points) Show that $X_t$ is a strictly stationary time series.

(b) (5 points) Show that the expected value of the zero crossing rate is given by $E(r) = 2p(1 - p_{11})$.

(c) (10 points) Now suppose $Z_t$ for any integer $t$ is given by

\[
Z_t = A \cos(\omega t) + B \sin(\omega t)
\]

where $A$ and $B$ are uncorrelated random variables with zero mean and variance $\sigma^2$, and $\omega$ is a fixed (non-random) frequency. Let $\rho_1 = E[Z_tZ_{t-1}]/\sigma^2$. Show that

\[
\rho_1 = \cos \left( \frac{\pi E[D]}{N-1} \right).
\]

Also show that $\rho_1 = \cos(\omega)$, and therefore we have a simple way to estimate the frequency $\omega$ from the zero-crossings.
(d) (5 points) Suppose that $Z_t$ is a zero-mean stationary Gaussian process with autocorrelation $R_Z(k)$. Define $\rho_1 = E[Z_t Z_{t-1}]/\sigma^2$, and show

$$\rho_1 = \cos \left( \frac{\pi E[D]}{N - 1} \right).$$

In the case $Z_t$ is a white noise process, show

$$\frac{E[D]}{N - 1} = \frac{1}{2}.$$