1. Suppose a random integer \( x \) is chosen. What is the probability that \( x^3 \) ends in 11?

This was probably the hardest problem on the exam. The key is to realize that when an integer \( x \) is cubed, only the last two digits of \( x \) affect the last two digits of \( x^3 \). Now, suppose we write \( x = \cdots rs = \cdots + 10r + s \), where \( r \) and \( s \) are the last two digits of \( x \), which are randomly chosen from \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Then, since only \( r \) and \( s \) affect the last two places of \( x^3 \), we may look at \((10r + s)^3\). Expanding \((10r + s)^3\), we see that only \( 30s^2r + s^3 \) can contribute to the last two digits, and thus we need only look \((r, s)\) solutions to \( 30s^2r + s^3 \) ending in 11. Clearly, the \( s^3 \) term is the only term that effects the last place, and only \( s = 1 \) can give a 1 in the last place of \( x^3 \). Now it remains to find integers \( r \) such that \( 30r + 1 \) has a 1 in the tens place. Only \( r = 7 \) does this. Therefore only the pair \((r, s) = (7, 1)\) can serve as the last two places of \( x \) in order for \( x^3 \) to end in 11. This occurs with probability \( 1/100 \).

2. How many ways are there to uniquely rearrange the letters of the word:

\[
HELLO
\]

(Hint: The L’s are considered identical.)

The L’s are identical, and therefore must be treated differently from the H, E, and O. The way to approach this problem is to think of having 5 boxes, and then count how many ways you can place the H, E, and O in those 5 boxes (the L’s will be put in the remaining boxes and it doesn’t matter how you do this since they are identical). This can be done in \( 5 \cdot 4 \cdot 3 = 60 \) ways.

3. Find the determinant of the matrix

\[
\begin{pmatrix}
15 & 3 & 1 & 0 \\
3 & 16 & 6 & -2 \\
1 & 6 & 4 & 1 \\
0 & -2 & 1 & 3 \\
\end{pmatrix}
\]

To solve this perform cofactor expansion. If the above matrix is \( A \), then

\[
det(A) = 15C_{11} - 3C_{12} + C_{13} - 0C_{14}
\]

where \( C_{ij} \) are the cofactors of \( A \). \( C_{11} = 28, C_{12} = 13 \) and \( C_{13} = 16 \). Thus \( det(A) = 397 \).

4. Two points \( P_1 \) and \( P_2 \) are randomly, and independently selected from the \((X, Y)\) plane according to the distribution

\[
f_{XY}(x, y) = \begin{cases} 
\frac{1}{4\pi^2} & (x, y) \in [-L, L] \times [-L, L] \\
0 & \text{elsewhere}
\end{cases}
\]

What is the probability that the line segment \( P_1P_2 \) lies entirely within the region \( \Omega \) given by

\[
\Omega = \left\{(x, y) : x^2 + y^2 \leq L^2 \right\}
\]

The main observation to make is that both points \( P_1 \) and \( P_2 \) must lie within the region \( \Omega \) in order for the line \( P_1P_2 \) to lie within \( \Omega \). The probability that \( P_1 \) lies within \( \Omega \) is \( Area(\Omega)/4L^2 = \pi/4 \). Since the selection of the points are independent, the probability that both lie within \( \Omega \) is \( \pi^2/16 \).