Solutions for Homework 6

4.2

a) \[ P_{\text{succ}} = Q_a(1,n)Q_r(0,n) + Q_a(0,n)Q_r(1,n) \]

Using Eqs(4.1) and (4.2),
\[
P_{\text{succ}} = \left[ (m-n)(1-q_a)^{m-n-1}q_a \right](1-q_r)^n + (1-q_a)^{m-n}[n(1-q_r)^n-1q_r] \\
= (1-q_a)^{m-n}(1-q_r)^n \left[ \frac{q_a}{1-q_a} + n \frac{q_r}{1-q_r} \right]
\]

b) Approximate \( q_a/(1-q_a) \) by \( q_a \) in the bracketed expression above. This is a good approximation for \( q_a \) small, whereas we cannot similarly approximate \( (1-q_a)^{m-n} \) by 1 since \( m-n \) might be large. We also approximate \( q_r/(1-q_r) \) by \( q_r \). Using the approximation \( (1-x)^y = e^{-xy} \), we then get
\[
P_{\text{succ}} = \exp[-(m-n)q_a \cdot nq_r] \left\{ (m-n)q_a + nq_r \right\} = G(n)e^{-G(n)}
\]

c) \( (1-x)^y = \exp[y \ln(1-x)] = \exp[y (-x - x^2/2 - x^3/3 ...)] \)
\[
= \exp(-yx) \exp[-x^2y/2 - x^3y/3 ...]
\]

which is equivalent to the desired relation. Note that for the approximation to be close (in terms of percentage error), \( x^2y \) must be close to 0.
4.7

a) Note that one packet successfully leaves the system each slot in which one or more packets are transmitted. Thus if all waiting packets attempt transmission in every slot, a successful transmission occurs in every slot in which packets are waiting. Since the expected delay is independent of the order in which packets are successfully transmitted (since each packet requires one slot), we see that the expected delay is the same as that of a centralized slotted FCFS system. Now compare this policy with an arbitrary policy for transmitting waiting packets; assume any given sequence of packet arrival times. Each time the arbitrary policy fails to attempt a transmission in a slot with waiting packets, the FCFS system (if it has waiting packets) decreases the backlog by 1 while the other policy does not decrease the backlog. Thus the backlog for the arbitrary system is always greater than or equal to that of the FCFS system (a formal proof of this would follow by induction on successive slots). Thus, by Little’s relation, the arbitrary system has an expected delay at least as great as the FCFS system.

b) This is just the slotted FDM system of section 3.5.1 with $m=1$ (i.e., a slotted $M/D/1$ queueing system). From Eq. (3.58), the queueing delay is $1/[2(1-\lambda)]$ slot times. The total delay, including service time, is then $1 + 1/[2(1-\lambda)]$.

c) The solution to b) can be rewritten as $1 + 1/2 + \lambda/[2(1-\lambda)]$ where the first term is the transmission time (i.e., 1 slot), the second term is the waiting time from an arrival to the beginning of a slot, and the third term is the delay due to collisions with other packets. If each subsequent attempt after an unsuccessful attempt is delayed by $k$ slots, this last term is multiplied by $k$. Thus the new total delay is $3/2 + k\lambda/[1-\lambda]$.

4.8

a) Let $X$ be the time in slots from the beginning of a backlogged slot until the completion of the first success at a given node. Let $q = q_1 p$ and note that $q$ is the probability that the node will be successful at any given slot given that it is still backlogged. Thus

$$P(X=i) = q(1-q)^{i-1} ; i \geq 1$$

$$E(X) = \sum_{i=1}^{\infty} iq(1-q)^{i-1} = \frac{1}{q}$$
The above summation uses the identity
\[
\sum_{i=1}^{\infty} i z^{i-1} = \sum_{i=1}^{\infty} \frac{dz^i}{dz} = \frac{d}{dz} \sum_{i=1}^{\infty} z^i = \frac{d[z/(1-z)]}{dz} = \frac{1}{(1-z)^2}
\]

Taking \( q = 1-z \) gives the desired result. A similar identity needed for the second moment is
\[
\sum_{i=1}^{\infty} i^2 z^{i-1} = \sum_{i=1}^{\infty} \frac{d^2 z^{i+1}}{dz^2} = \sum_{i=1}^{\infty} \frac{dz^i}{dz} = \frac{d^2 [z^2/(1-z)]}{dz^2} - \frac{1}{(1-z)^2} = \frac{1+z}{(1-z)^3}
\]

Using this identity with \( q=1-z \), we have
\[
E(X^2) = \sum_{i=1}^{\infty} i^2 q(1-q)^{i-1} = \frac{2-q}{q^2} = \frac{2-pq}{(pq)^2}
\]

b) For an individual node, we have an M/G/1 queue with vacations. The vacations are deterministic with a duration of 1 slot, and the service time has the first and second moments found in part a). Thus, using Eq. (3.55) for the queueing delay and adding an extra service time to get the system delay,
\[
T = \frac{(\lambda/m) E(X^2)}{2(1-p)} + \frac{1}{2} + \frac{1}{q} = \frac{\lambda(2-p)}{2q^2 (1-p)m} + \frac{1}{2} + \frac{1}{q}
\]

Since the arrival rate is \( \lambda/m \) and the service rate is \( q \), we have \( \rho = \lambda/(mq) \). Substituting this into the above expression for \( T \) and simplifying,
\[
T = \frac{1}{q_p(p-1) + \frac{1-2p}{2(1-p)}}
\]

c) For \( p=1 \) and \( q_p=1/m \), we have \( \rho = \lambda \), so that
\[
T = \frac{m}{1-\lambda} + \frac{1-2\lambda}{2(1-\lambda)}
\]

In the limit of large \( m \), this is twice the delay of TDM as given in Eq. (3.59).

4.9

a) Let \( \nu \) be the mean number of packets in the system. Given \( n \) packets in the system, with each packet independently transmitted in a slot with probability \( \nu^{-1} \), the probability of an idle slot, \( P[I|n] \) is \( (1-\nu^{-1})^n \). The joint probability of an idle slot and \( n \) packets in the system is then
\[
P(n,I) = P(n)P[I|n] = \frac{\exp(-\nu)\nu^n}{n!} (1-\nu^{-1})^n
\]
\[ P(I) = \sum_{n=0}^{\infty} P(n,I) = \sum_{n=0}^{\infty} \frac{\exp(-v)(v-1)^n}{n!} = \frac{1}{e} \]

b) Using the results above, we can find \( P(n \mid I) \)

\[ P(n \mid I) = \frac{P(n,I)}{P(I)} = \frac{\exp(-v+1)(v-1)^n}{n!} \]

Thus, this probability is Poisson with mean \( v-1 \).

c) We can find the joint probability of success and \( n \) in the system similarly

\[ P(n,S) = P(n)P(S \mid n) = \frac{\exp(-v)v^n}{n!} n(1-v^{-1})^{n-1}v^{-1} = \frac{\exp(-v)(v-1)^{n-1}}{(n-1)!} \]

\[ P(S) = \sum_{n=0}^{\infty} \frac{\exp(-v)(v-1)^{n-1}}{(n-1)!} = \frac{1}{e} \]

d) From this, the probability that there were \( n \) packets in the system given a success is

\[ P(n \mid S) = \frac{P(n,S)}{P(S)} = \frac{\exp(-v+1)(v-1)^{n-1}}{(n-1)!} \]

Note that \( n-1 \) is the number of remaining packets in the system with the successful packet removed, and it is seen from above that this remaining number is Poisson with mean \( v-1 \).

4.10

a) All nodes are initially in mode 2, so when the first success occurs, the successful node moves to mode 1. While that node is in mode 1, it transmits in every slot, preventing any other node from entering mode 1. When that node eventually transmits all its packets and moves back to mode 2, we return to the initial situation of all nodes in mode 2. Thus at most one node at a time can be in mode 1.

b) The probability of successful transmission, \( p_1 \), is the probability that no other node is transmitting. Thus \( p_1 = (1-q_2)^{m-1} \). The first and second moment of the time between successful transmissions is the same computation as in problem 4.8a. We have

\[ \bar{X} = \frac{1}{p_1} \quad \bar{X}^2 = \frac{2-p_1}{p_1^2} \]

c) The probability of some successful dummy transmission in a given slot when all nodes are in mode 2 is \( p_2 = m q_4 (1-q_2)^{m-1} \). The first two moments of the time to such a success is the same problem as above, with \( p_2 \) in place of \( p_1 \). Thus

\[ \bar{\nu} = \frac{1}{p_2} \quad \bar{\nu}^2 = \frac{2-p_2}{p_2^2} \]
d) The system is the same as the exhaustive multiuser system of subsection 3.5.2 except for the random choice of a new node to be serviced at the end of each reservation interval. Thus for the $i$th packet arrival to the system as a whole, the expected queueing delay before the given packet first attempts transmission is

$$E(W_i) = E(R_i) + E(N_i)\tilde{X} + E(Y_i)$$

where $R_i$ is the residual time to completion of the current packet service or reservation interval and $Y_i$ is the duration of all the whole reservation intervals during which packet $i$ must wait before its node enters mode 1. Since the order of serving packets is independent of their service time, $E(N_i) = \lambda E(W)$ in the limit as $i$ approaches infinity. Also, since the length of each reservation interval is independent of the number of whole reservation intervals that the packet must wait, $E(Y_i)$ is the expected number of whole reservation intervals times the expected length of each. Thus

$$W = \frac{R + E(S)\tilde{v}}{1-\rho}, \quad \rho = \lambda \tilde{X}$$

e) As in Eq. (3.64),

$$R = \frac{\lambda X^2}{2} + \frac{(1-\rho)\tilde{v}^2}{2\tilde{v}}$$

Finally, the number of whole reservation intervals that the packet must wait is zero with probability $1/m$, one with probability $(1-1/m)/m$, and in general $i$ with probability $(1-1/m)^i/m$. Thus $E(S) = m-1$. Substituting these results and those of parts b) and c) into the above expression for $W$, we get the desired expression.
4.19

a) Consider first the expected number of successes on all but the final slot of a CRP and then the expected number on the final slot. Each non-final success occurs on a left interval, and, in terms of the Markov chain of Figure 4.13, corresponds to a transition from a left interval (top row of states) to a right interval (bottom row of states). Thus, for any given CRP, the number of non-final successes is the number of transitions from left to right states. Since the right states (i\geq1) are entered only by these transitions, the number of non-final successes is the number of visits to right states (other than the state (R,0)). Thus the expected number of non-final successes is the expected number of visits to right states, i\geq1. Finally note that, except when the CRP consists only of a single idle slot (which occurs with probability \exp(-G_0)), the final slot of a CRP is a success. Thus

\[ \bar{n} = 1 - \exp(-G_0) + \sum_{i=1}^{\infty} p(R,i) \]

b) Whenever a collision occurs in a left interval, the corresponding right interval is returned to the waiting interval. The number of packets successfully transmitted in a CRP is the number in the original allocation interval less the number returned by the mechanism above. Thus the expected number transmitted is the expected number in the original interval less the expected number returned; this is true despite statistical dependencies between the original number and the returned number. Given that an interval of size x is returned, the number of packets in the returned interval is Poisson with mean \lambda x, independent of the past history of the CRP. Thus, given that a fraction f of the interval is returned (via one or more intervals), the expected number of returns is \lambda x f. Averaging over f (which is dependent on the number in the original interval), the expected number of returns is \lambda x_0 E(f). Thus

\[ \bar{n} = \lambda x_0 [1 - E(f)] \]
4.31

a) A simultaneous transmission on links 1 and 3 causes a collision for the transmission on link 1; similarly, a collision on link 2 occurs if 2 and 3 are used simultaneously. Finally, simultaneous transmissions on links 1 and 2 cause a collision for both transmissions. Thus at most one link can be used successfully at a time and \( f_1 + f_2 + f_3 \leq 1 \). To view this in terms of Eq. (4.75), we let \( x_1, x_2, \) and \( x_3 \) be the collision free vectors \((100), (010),\) and \((001)\) respectively, and we let \( x_4 \) be the trivial CFV \((000)\). Then, for \( 1 \leq i \leq 3 \), \( f_i \) corresponds to \( a_i \) in Eq. (4.75) and the constraints \( a_i \geq 0 \) and \( a_1 + a_2 + a_3 + a_4 = 1 \) is equivalent to \( f_1 + f_2 + f_3 \leq 1 \).

b) From Eq. (4.77), we have

\[
\begin{align*}
p_1 &= (1-q_3)(1-q_2) \\
p_2 &= (1-q_3)(1-q_1) \\
p_3 &= 1
\end{align*}
\]

Eq. (4.78) then gives us the fractional utilizations

\[
\begin{align*}
f &= f_1 = q_1(1-q_3)(1-q_2) & (i) \\
f &= f_2 = q_2(1-q_3)(1-q_1) & (ii) \\
2f &= f_3 = q_3 & (iii)
\end{align*}
\]

Taking the ratio of (ii) and (i),

\[
1 = \frac{q_2 (1-q_1)}{q_1 (1-q_2)} ; \quad \text{thus } q_1 = q_2
\]

c) Using \( q_1 = q_2 \) and \( q_3 = 2f \) in (i) above, we have \( f = q_1(1-2f)(1-q_1) \). Thus

\[
\frac{f}{1-2f} = q_1(1-q_1) \leq \frac{1}{4}
\]

The inequality above follows by taking the maximum of \( q_1(1-q_1) \) over \( q_1 \) between 0 and 1. It follows from this that \( f \leq 1/6 \).