Solution for Homework 5

3.50

(a) Since \( \{p_j\} \) is the stationary distribution, we have for all \( j \in \mathcal{S} \)

\[
p_j \left( \sum_{k=1}^{\mathcal{S}} q_{kj} + \sum_{k=1}^{\mathcal{S}} q_{jk} \right) = \sum_{k=1}^{\mathcal{S}} p_k q_{kj} + \sum_{k=1}^{\mathcal{S}} p_k q_{kj}
\]

Using the given relation, we obtain for all \( j \in \mathcal{S} \)

\[
p_j \sum_{k=1}^{\mathcal{S}} q_{kj} = \sum_{k=1}^{\mathcal{S}} p_k q_{kj}
\]

Dividing by \( \sum_{k=1}^{\mathcal{S}} p_k \), it follows that

\[
\bar{p}_j \sum_{k=1}^{\mathcal{S}} q_{kj} = \sum_{k=1}^{\mathcal{S}} \bar{p}_k q_{kj}
\]

for all \( j \in \mathcal{S} \), showing that \( \{\bar{p}_j\} \) is the stationary distribution of the truncated chain.

(b) If the original chain is time reversible, we have \( p_j q_{ji} = p_i q_{ij} \) for all \( i \) and \( j \), so the condition of part (a) holds. Therefore, we have \( \bar{p}_j q_{ji} = \bar{p}_i q_{ij} \) for all states \( i \) and \( j \) of the truncated chain.

(c) The finite capacity system is a truncation of the two independent M/M/1 queues system, which is time reversible. Therefore, by part (b), the truncated chain is also time reversible. The formula for the steady state probabilities is a special case of Eq. (3.39) of Section 3.4.

3.52

Consider a customer arriving at time \( t_1 \) and departing at time \( t_2 \). In reversed system terms, the arrival process is independent Poisson, so the arrival process to the left of \( t_2 \) is independent of the times spent in the system of customers that arrived at or to the right of \( t_2 \). In particular, \( t_2 - t_1 \) is independent of the (reversed system) arrival process to the left of \( t_2 \). In forward system terms, this means that \( t_2 - t_1 \) is independent of the departure process to the left of \( t_2 \).
The session numbers and their rates are shown below:

<table>
<thead>
<tr>
<th>Session</th>
<th>Session number p</th>
<th>Session rate $x_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACE</td>
<td>1</td>
<td>100/60 = 5/3</td>
</tr>
<tr>
<td>ADE</td>
<td>2</td>
<td>200/60 = 10/3</td>
</tr>
<tr>
<td>BCEF</td>
<td>3</td>
<td>500/60 = 25/3</td>
</tr>
<tr>
<td>BDEF</td>
<td>4</td>
<td>600/60 = 30/3</td>
</tr>
</tbody>
</table>

The link numbers and the total link rates calculated as the sum of the rates of the sessions crossing the links are shown below:

<table>
<thead>
<tr>
<th>Link</th>
<th>Total link rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>$x_1 = 5/3$</td>
</tr>
<tr>
<td>CE</td>
<td>$x_1 + x_3 = 30/3$</td>
</tr>
<tr>
<td>AD</td>
<td>$x_2 = 10/3$</td>
</tr>
<tr>
<td>BD</td>
<td>$x_4 = 10$</td>
</tr>
<tr>
<td>DE</td>
<td>$x_2 + x_4 = 40/3$</td>
</tr>
<tr>
<td>BC</td>
<td>$x_3 = 25/3$</td>
</tr>
<tr>
<td>EF</td>
<td>$x_3 + x_4 = 55/3$</td>
</tr>
</tbody>
</table>

For each link $(i,j)$ the service rate is
\[
\mu_{ij} = \frac{500000}{1000} = 50 \text{ packets/sec},
\]
and the propagation delay is $D_{ij} = 2 \times 10^{-3}$ secs. The total arrival rate to the system is
\[
\gamma = \sum_i x_i = \frac{5}{3} + \frac{10}{3} + \frac{25}{3} + \frac{30}{3} = \frac{70}{3}
\]
The average number on each link $(i, j)$ (based on the Kleinrock approximation formula) is:
\[
N_{ij} = \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}} + \lambda_{ij}D_{ij}
\]

From this we obtain:

<table>
<thead>
<tr>
<th>Link</th>
<th>Average Number of Packets on the Link</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>$(5/3)/(150/3 - 5/3) + (5/3)(2/1000) = 5/145 + 1/300$</td>
</tr>
<tr>
<td>CE</td>
<td>$1/4 + 1/50$</td>
</tr>
<tr>
<td>AD</td>
<td>$1/14 + 1/150$</td>
</tr>
<tr>
<td>BD</td>
<td>$1/4 + 1/50$</td>
</tr>
</tbody>
</table>
The average total number in the system is \( N = \sum_{(i,j)} N_{ij} \equiv 1.84 \) packet. The average delay over all sessions is \( T = N/\gamma = 1.84 \times (3/70) = 0.0789 \) secs. The average delay of the packets of an individual session are obtained from the formula

\[
T_p = \sum_{(i,j) \in p} \left[ \frac{\lambda_{ij}}{\mu_{ij}(\mu_{ij} - \lambda_{ij})} + \frac{1}{\mu_{ij}} + D_{ij} \right]
\]

For the given sessions we obtain applying this formula

<table>
<thead>
<tr>
<th>Session</th>
<th>p</th>
<th>Average Delay ( T_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.050</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.053</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.087</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.090</td>
</tr>
</tbody>
</table>
We have \( \sum_{i=0}^{m} p_i = 1 \).

The arrival rate at the CPU is \( \lambda/p_0 \) and the arrival rate at the \( i \)th I/O port is \( \lambda_i/p_i \). By Jackson's Theorem, we have

\[
P(n_0, n_1, \ldots, n_m) = \prod_{i=0}^{m} p_i^{n_i} (1 - p_i)
\]

where \( p = \frac{\lambda}{\mu_0 p_0} \)

and \( p_i = \frac{\lambda_i}{\mu_i p_0} \) for \( i > 0 \)

The equivalent tandem system is as follows:

![Diagram of tandem system](image)

The arrival rate is \( \lambda \). The service rate for queue 0 is \( \mu_0 p_0 \) and for queue \( i (i > 0) \) is \( \mu_i p_0/p_i \).

Let \( \lambda_0 \) be the arrival rate at the CPU and let \( \lambda_i \) be the arrival rate at I/O unit \( i \). We have

\[
\lambda_i = p_i \lambda_0, \quad i = 1, \ldots, m.
\]

Let

\[
\bar{\lambda}_0 = 1, \quad \bar{\lambda}_i = p_i, \quad i = 1, \ldots, m,
\]

and

\[
p_i = \frac{\bar{\lambda}_i}{\mu_i}, \quad i = 0, 1, \ldots, m.
\]

By Jackson's Theorem, the occupancy distribution is

\[
P(n_0, n_1, \ldots, n_m) = \frac{p_0^{n_0} p_1^{n_1} \cdots p_m^{n_m}}{G(M)},
\]

where \( G(M) \) is the normalization constant corresponding to \( M \) customers,

\[
G(M) = \sum_{n_0, n_1, \ldots, n_m} p_0^{n_0} p_1^{n_1} \cdots p_m^{n_m}
\]
\[ U_0 = \frac{\lambda_0}{\mu_0} \]

be the utilization factor of the CPU. We have

\[ U_0 = P(n_0 \geq 1) = \sum_{n_1, \ldots, n_m} P(n_0, n_1, \ldots, n_m) \]

\[ = \sum_{n_1, \ldots, n_m} \rho_0 \rho_1^{n_1} \cdots \rho_m^{n_m} \frac{G(M)}{G(M)} \]

\[ = \rho_0 \frac{G(M-1)}{G(M)} = \frac{1}{\mu_0} \frac{G(M-1)}{G(M)}, \]

where we used the change of variables \( n'_0 = n_0 - 1 \). Thus the arrival rate at the CPU is

\[ \lambda_0 = \frac{G(M-1)}{G(M)} \]

and the arrival rate at the I/O unit i is

\[ \lambda_i = \frac{p_i G(M-1)}{G(M)}, \quad i = 1, \ldots, m. \]

3.64

(a) The state is determined by the number of customers at node 1 (one could use node 2 just as easily). When there are customers at node 1 (which is the case for states 1, 2, and 3), the departure rate from node 1 is \( \mu_1 \); each such departure causes the state to decrease as shown below. When there are customers in node 2 (which is the case for states 0, 1, and 2), the departure rate from node 2 is \( \mu_2 \); each such departure causes the state to increase.

(b) Letting \( p_i \) be the steady state probability of state i, we have \( p_i = p_{i-1} \rho \), where \( \rho = \mu_2/\mu_1 \). Thus \( p_i = p_0 \rho^i \). Solving for \( p_0 \),

\[ p_0 = \frac{[1 + \rho + \rho^2 + \rho^3]^{-1}}{\rho}, \quad p_i = p_0 \rho^i, \quad i = 1, 2, 3. \]

(c) Customers leave node 1 at rate \( \mu_1 \) for all states other than state 0. Thus the time average rate \( r \) at which customers leave node 1 is \( \mu_1(1 - P_0) \), which is

\[ r = \frac{\rho + \rho^2 + \rho^3}{1 + \rho + \rho^2 + \rho^3} \mu_1 \]

(d) Since there are three customers in the system, each customer cycles at one third the rate at which departures occur from node 1. Thus a customer cycles at rate \( r/3 \).

(e) The Markov process is a birth-death process and thus reversible. What appears as a departure from node i in the forward process appears as an arrival to node i in the backward process. If we order the customers 1, 2, and 3 in the order in which they depart a node, and note that this order never changes (because of the FCFS service at each node), then we see that in the backward process, the customers keep their identity, but the order is reversed with backward departures from node i in the order 3, 2, 1, 3, 2, 1, ... .