This is an 180 minute exam. Please answer the following questions in the notebooks provided. This is a closed book test. Make sure that you have included your name, personal 4 digit code (unrelated to your RU ID digits) and signature in each book used (5 points). Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. 20 points Let \( X_1, Z_1, Z_2, \ldots \) be iid Bernoulli random variables which take values 0 and 1 with equal probability. Define the sequence of random variables \( X_i \) as

\[
X_{i+1} = X_i + Z_i, \quad i = 1, 2, \ldots, n - 1.
\]

Find the mutual information \( I(X_1; X_2, X_3, \ldots, X_n) \).

2. 35 points Let \( Z \) take values 0 and 1 with probabilities \( 1 - p \) and \( p \). Let \( X \), which is independent of \( Z \), take values \( 1, 2, \ldots, n \) with probabilities \( q = [q_1, q_2, \ldots, q_n] \). Let \( Y = XZ \).

(a) 10 points Find the entropy of \( Y \) in terms of the entropies of \( X \) and \( Z \).

(b) 10 points Find the \( p \) and \( q \) that maximize \( H(Y) \).

(c) 15 points Suppose \( X \) and \( Y \) are the input and output of a discrete memoryless channel. For fixed \( p \), what is the capacity \( C(p) \) of the channel? What value of \( p \) maximizes \( C(p) \)?

3. 40 points A discrete memoryless multiple access channel has inputs \( X_1 \) and \( X_2 \) and output \( Y = X_1 + X_2 \). The inputs \( X_1 \) and \( X_2 \) both use alphabet \( \mathcal{X} = \{0, 1, 2\} \); the output \( Y \) has alphabet \( \mathcal{Y} = \{0, 1, \ldots, 4\} \).

(a) 20 points Under the assumption that each \( X_i \) uses equiprobable inputs, find and sketch the \( (R_1, R_2) \) region of achievable rates for this 2-user MAC.

(b) 20 points Suppose user 1 and user 2 collaborate and act as single transmitter of rate \( R \) with input \( X = (X_1, X_2) \) and output \( Y \). What is the capacity of the channel? What input distribution achieves capacity?

4. 20 points Consider two parallel channels with independent Gaussian noise \( Z_1 \) and \( Z_2 \) with variances \( N_1 = 1 \) and \( N_2 = 2 \). The signalling is

\[
Y_1 = X_1 + Z_1 \\
Y_2 = X_2 + Z_2
\]

The transmitter is subject to the power constraint \( E[X_1^2 + X_2^2] \leq P \). Find and sketch the capacity \( C(P) \) of this channel as a function of \( P \).
5. 55 points Every coding theorem we proved this semester included a converse that was proven using the Fano bound. For example, in the case of a discrete, memoryless channel, for any sequence of \((2^nR,n)\) codes with message index \(X\), codewords \(X^n(W)\), receiver output \(Y^n\), decoding function \(g(Y^n)\), and error probability \(P_e^{(n)} = P [W \neq g(Y^n)]\), the proof used these steps:

\[
\begin{align*}
nR &= H(W) \\
&= H(W|Y^n) + I(W;Y^n) \\
&\leq H(W|Y^n) + I(X^n(W);Y^n) \\
&\leq 1 + P_e^{(n)} nR + I(X^n(W);Y^n) \\
&\leq 1 + P_e^{(n)} nR + nC
\end{align*}
\]

(a) 25 points For each of the above steps, (1) through (5), there is a specific reason that step holds. Given a precise justification for each of the five steps above.

(b) 10 points Explain how step (5) implies a converse to the coding theorem.

(c) 20 points For one of the above five steps, the correct reason is simply “the Fano bound” or “Fano’s inequality.” Derive the Fano bound as used in the above five step proof. Hint: The proof defines the error event

\[
E = \begin{cases} 
1 & g(Y^n) \neq W \\
0 & g(Y^n) = W
\end{cases}
\]

and then expands \(H(E,W|Y^n)\) in two different ways.