This is an 80 minute exam. You may have an additional 100 minutes to answer the following five questions in the notebooks provided. You are permitted 2 double-sided sheet of notes. Make sure that you have included your name, ID number (last 4 digits only) and signature in each book used (5 points). Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. 20 points Let $Y = g(X)$ be deterministic function of discrete random variable $X$.
   (a) 5 points Give an example of a random variable $X$ and a function $Y = g(X)$ such that $H(Y) < H(X)$.
   (b) 5 points Give an example of a random variable $X$ and a function $Y = g(X)$ such that $H(Y) = H(X)$.
   (c) 10 points Either give an example of a random variable $X$ and a function $Y = g(X)$ such that $H(Y) \geq H(X)$ or prove that $H(Y) \leq H(X)$. Hint: look at $H(X, Y)$.

2. 20 points Let be random variables such that $X \rightarrow Y \rightarrow Z$, $Y \rightarrow Z \rightarrow X$, $Z \rightarrow X \rightarrow Y$. If $I(X; Y) = 3$, find $I(X; Z)$ and $I(Y; Z)$. Or, if these quantities cannot be known, find tight upper and lower bounds. Make sure to justify your answers.

3. 15 points Consider the code $\{0, 01\}$. Justify your answers to the following questions:
   (a) 5 points Is the code instantaneous?
   (b) 5 points Is the code nonsingular?
   (c) 5 points Is the code uniquely decodable?

4. 10 points The source coding theorem shows that the optimal source code for random variable $X$ has expected length $L \leq H(X) + 1$. Find an example of a random variable $X$ for which the optimal code has $L > H(X) + 1 - \epsilon$ for any small $\epsilon > 0$. Make sure you justify your answer.

5. 30 points A source has an alphabet of 4 letters, $a_1, a_2, a_3, a_4$ with probabilities $p_1 \geq p_2 \geq p_3 \geq p_4$.
   (a) Suppose $p_1 > p_2 = p_3 = p_4$. Find the smallest number $q$ such that $p_1 > q$ implies $n_1 = 1$, where $n_1$ is the length of the code word for $a_1$ in a binary Huffman code for the source.
   (b) Show by example that if $p_1 = q$ (your answer in part (a)), then a Huffman code exists with $n_1 > 1$.
   (c) Now assume the more general condition $p_1 > p_2 \geq p_3 \geq p_4$. Does $p_1 > q$ still imply that $n_1 = 1$? Why or why not?
   (d) Now assume that the source has an arbitrary number $K$ of letters, with $p_1 > p_2 \geq \cdots \geq p_K$. Does $p_1 > q$ now imply that $n_1 = 1$? Explain.
   (e) 20 points Now assume the source has $K$ letters $a_1, \ldots, a_K$, with $p_1 \geq p_2 \geq \cdots \geq p_K$. Find the largest number $q'$ such that $p_1 < q'$ implies that $n_1 > 1$.

Announcement: Class on Thursday March 13 will be canceled. Instead, we will have class on Thursday March 20 (during Spring break) at the usual hour.