1. 25 points Each day, it rains (event $R = 1$) or not (event $R = 0$). A TV station subscribes to a weather forecasting service which delivers a prediction: $Q = 1$ if the prediction is rain, or $Q = 0$ if no rain. Each day, the TV weatherman makes the weather announcement $A = Q$. Fortunately, $Q$ and $R$ are not independent and have the following PMF

<table>
<thead>
<tr>
<th>$P_{R,Q}(r,q)$</th>
<th>$q = 1$</th>
<th>$q = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 1$</td>
<td>$1/8$</td>
<td>$1/16$</td>
</tr>
<tr>
<td>$r = 0$</td>
<td>$3/16$</td>
<td>$10/16$</td>
</tr>
</tbody>
</table>

(a) 10 points A student observes that the weatherman is correct with probability $12/16$ but could be correct with probability $13/16$ by always making the weather announcement $A = 0$, corresponding to “no rain.” The student applies for the weatherman’s job, but the boss, who is an information theorist, turns him down. Why?

(b) 10 points The prediction $Q$ is based on a maximum likelihood (ML) hypothesis test (using some unspecified observations $X$) as to whether $R = 0$ or $R = 1$. For what values of $p = P(R = 1)$ does the weatherman’s announcement $A = Q$ also maximize the probability $P(C)$ of a correct prediction based on $Q$?

(c) 5 points Was it necessary in the preceding step to specify that the prediction $Q$ was based on maximum likelihood?

2. 30 points Let $X, Y, Z$ be an ensemble of discrete random variables. In each of the following problems, there exists an equality or inequality between the two quantities. Fill in the blank $\_\_\_$ with the appropriate relationship ($\leq$, $=$, or $\geq$) and justify the correctness of that relationship.

(a) $I(X, Y; Z) \_\_\_\_ I(X; Z)$

(b) $H(X|Z) \_\_\_\_ H(X, Y|Z)$

(c) $H(X, Z) - H(X) \_\_\_\_ H(X, Y, Z) - H(X, Y)$

3. 30 points The process $X_1, X_2, \ldots$ is an iid Bernoulli ($p$) random sequence. Let $R_n = (X_1 + \cdots + X_n)/n$ denote the success rate of the process.

(a) In terms of $R_n$, characterize the set $A^{(n)}_\epsilon$ of typical sequences.

(b) When $p = 1/2$, what sequences are typical?

(c) For what values of $p > 1/2$, if any, does $A^{(n)}_\epsilon$ include the most probable sequence? For such $p$, does $A^{(n)}_\epsilon$ include the most probably sequence for all $\epsilon > 0$?

4. 20 points Consider the code $\{0, 10, 01\}$ for a ternary source. Justify your answers to the following questions:

(a) 5 points Is the code instantaneous?

(b) 5 points Is the code nonsingular?

(c) 5 points Is the code uniquely decodable?
5. **50 points** The outcome of a roulette wheel is either red \( X = 1 \) or black \( X = 0 \), equiprobably and independently from spin to spin. By observing the ball until the last instant that bets can be placed, a gambler can predict \( X \) with some accuracy. Given the gambler’s prediction, \( Y = 0 \) or \( Y = 1 \), conditional probabilities for \( X \) are given by

\[
P_{X|Y}(1|1) = P_{X|Y}(0|0) = \frac{3}{4}.
\]

(a) Calculate the mutual information \( I(X; Y) \).

(b) The gambler has some initial capital \( C_0 \). On each spin, she bets a fraction \( 1 - q \) of her total capital on the predicted color and a fraction \( q \) on the other color. Let \( Z_n = 1 \) if the gambler’s prediction is correct on trial \( n \). After \( N \) spins, the gamblers capital is the random variable \( C_N \). Express \( C_N \) in terms of \( Z_1, \ldots, Z_N \).

(c) Find \( q^*_C \), the value of \( q \) that maximizes the expected value \( E[C_N] \).

(d) Define the rate of growth as \( R_N = \frac{1}{N} \log_2 \frac{C_N}{C_0} \).

Find \( q^*_R \), the value of \( q \) that maximizes the expected value \( E[R_N] \). For \( q = q^*_R \), compare \( E[R_N] \) and \( I(X; Y) \).

(e) If you were the gambler, would you use \( q = q^*_R \) or \( q = q^*_C \)? Explain why.