Chapter 14

Network Information Theory

1. The cooperative capacity of a multiple access channel. (Figure 14.1)

Figure 14.1: Multiple access channel with cooperating senders.

(a) Suppose $X_1$ and $X_2$ have access to both indices $W_1 \in \{1, 2^n R_1\}, W_2 \in \{1, 2^n R_2\}$. Thus the codewords $X_1(W_1, W_2), X_2(W_1, W_2)$ depend on both indices. Find the capacity region.

(b) Evaluate this region for the binary erasure multiple access channel $Y = X_1 + X_2, X_i \in \{0, 1\}$. Compare to the non-cooperative region.

Solution: Cooperative capacity of multiple access channel

(a) When both senders have access to the pair of messages to be transmitted, they can act in concert. The channel is then equivalent to a single user channel with
input alphabet $\mathcal{X}_1 \times \mathcal{X}_2$, and a larger message set $W_1 \times W_2$. The capacity of this single user channel is $C = \max_{p(x)} I(X; Y) = \max_{p(x_1, x_2)} I(X_1, X_2; Y)$. The two senders can send at any combination of rates with the total rate

$$R_1 + R_2 \leq C \quad (14.1)$$

(b) The capacity for the binary erasure multiple access channel was evaluated in class. When the two senders cooperate to send a common message, the capacity is

$$C = \max_{p(x_1, x_2)} I(X_1, X_2; Y) = \max H(Y) = \log 3, \quad (14.2)$$

achieved by (for example) a uniform distribution on the pairs, (0,0), (0,1) and (1,1). The cooperative and non-cooperative regions are illustrated in Figure 14.2.

Figure 14.2: Cooperative and non-cooperative capacity for a binary erasure multiple access channel

2. Capacity of multiple access channels. Find the capacity region for each of the following multiple access channels:
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(a) Additive modulo 2 multiple access access channel. \( X_1 \in \{0, 1\}, X_2 \in \{0, 1\}, Y = X_1 \oplus X_2. \)

(b) Multiplicative multiple access channel. \( X_1 \in \{-1, 1\}, X_2 \in \{-1, 1\}, Y = X_1 \cdot X_2. \)

Solution: Examples of multiple access channels.

(a) Additive modulo 2 MAC.

\( Y = X_1 \oplus X_2. \) Quite clearly we cannot send at a total rate of more than 1 bit, since \( H(Y) \leq 1. \) We can achieve a rate of 1 bit from sender 1 by setting \( X_2 = 0, \) and similarly we can send 1 bit/transmission from sender 2. By simple time sharing we can achieve the entire capacity region which is shown in Figure 14.3.

(b) Multiplier channel.

\( X_1, X_2 \in \{-1, 1\}, Y = X_1 \cdot X_2. \)

This channel is equivalent to the previous channel with the mapping \(-1 \rightarrow 1\) and \(1 \rightarrow 0.\) Hence the capacity region is the same as the previous channel.

3. Cut-set interpretation of capacity region of multiple access channel. For the multiple access channel we know that \((R_1, R_2)\) is achievable if

\[
R_1 < I(X_1; Y \mid X_2), \tag{14.3}
\]

\[
R_2 < I(X_2; Y \mid X_1), \tag{14.4}
\]

\[
R_1 + R_2 < I(X_1, X_2; Y), \tag{14.5}
\]
which when combined with the power constraints, and taking the limit at $n \to \infty$, we obtain the desired converse, i.e.,

$$R_1 < \frac{1}{2} \log(1 + \frac{P_1}{N}),$$  \hspace{1cm} (14.68)

$$R_2 < \frac{1}{2} \log(1 + \frac{P_2}{N}),$$  \hspace{1cm} (14.69)

$$R_1 + R_2 < \frac{1}{2} \log(1 + \frac{P_1 + P_2}{N}).$$  \hspace{1cm} (14.70)

6. **Unusual multiple access channel.** Consider the following multiple access channel: $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0,1\}$. If $(X_1, X_2) = (0,0)$, then $Y = 0$. If $(X_1, X_2) = (0,1)$, then $Y = 1$. If $(X_1, X_2) = (1,0)$, then $Y = 1$. If $(X_1, X_2) = (1,1)$, then $Y = 0$ with probability $\frac{1}{2}$ and $Y = 1$ with probability $\frac{1}{2}$.

(a) Show that the rate pairs $(1,0)$ and $(0,1)$ are achievable.

(b) Show that for any non-degenerate distribution $p(x_1)p(x_2)$, we have $I(X_1, X_2; Y) < 1$.

(c) Argue that there are points in the capacity region of this multiple access channel that can only be achieved by timesharing, i.e., there exist achievable rate pairs $(R_1, R_2)$ which lie in the capacity region for the channel but not in the region defined by

$$R_1 \leq I(X_1; Y|X_2),$$  \hspace{1cm} (14.71)

$$R_2 \leq I(X_2; Y|X_1),$$  \hspace{1cm} (14.72)

$$R_1 + R_2 \leq I(X_1, X_2; Y)$$  \hspace{1cm} (14.73)

for any product distribution $p(x_1)p(x_2)$. Hence the operation of convexification strictly enlarges the capacity region. This channel was introduced independently by Csiszár and Körner[3] and Bierbaum and Wallmeier[2].

Solution:

**Unusual multiple access channel.**

(a) It is easy to see how we could send 1 bit/transmission from $X_1$ to $Y$—simply set $X_2 = 0$. Then $Y = X_1$, and we can send 1 bit/transmission to from sender 1 to the receiver.

Alternatively, if we evaluate the achievable region for the degenerate product distribution $p(x_1)p(x_2)$ with $p(x_1) = (\frac{1}{2}, \frac{1}{2})$, $p(x_2) = (1,0)$, we have $I(X_1; Y|X_2) = 1$, $I(X_2; Y|X_1) = 0$, and $I(X_1, X_2; Y) = 1$. Hence the point $(1,0)$ lies in the achievable region for the multiple access channel corresponding to this product distribution.

By symmetry, the point $(0,1)$ also lies in the achievable region.
(b) Consider any non-degenerate product distribution, and let \( p_1 = p(X_1 = 1) \), and let \( p_2 = p(X_2 = 1) \). By non-degenerate we mean that \( p_1 \neq 0 \) or 1, and \( p_2 \neq 0 \) or 1. In this case, \( Y = 0 \) when \( (X_1, X_2) = (0, 0) \) and half the time when \( (X_1, X_2) = (1, 1) \), i.e., with a probability \( (1 - p_1)(1 - p_2) + \frac{1}{2} p_1 p_2 \). \( Y_1 = 1 \) for the other input pairs, i.e., with a probability \( p_1 (1 - p_2) + p_2 (1 - p_1) + \frac{1}{2} p_1 p_2 \). We can evaluate the achievable region of the multiple access channel for this product distribution. In particular,

\[
R_1 + R_2 \leq I(X_1, X_2; Y) = H(Y) - H(Y | X_1, X_2) = H((1 - p_1)(1 - p_2) + \frac{1}{2} p_1 p_2) - p_1 p_2.
\]

(14.74)

Now \( H((1 - p_1)(1 - p_2) + \frac{1}{2} p_1 p_2) \leq 1 \) (entropy of a binary random variable is at most 1) and \( p_1 p_2 > 0 \) for a non-degenerate distribution. Hence \( R_1 + R_2 \) is strictly less than 1 for any non-degenerate distribution.

(c) The degenerate distributions have either \( R_1 \) or \( R_2 \) equal to 0. Hence all the distributions that achieve rate pairs \((R_1, R_2)\) with both rates positive have \( R_1 + R_2 < 1 \). For example the union of the achievable regions over all product distributions does not include the point \((\frac{1}{2}, \frac{1}{2})\). But this point is clearly achievable by timesharing between the points \((1, 0)\) and \((0, 1)\). Or equivalently, the point \((\frac{1}{2}, \frac{1}{2})\) lies in the convex hull of the union of the achievable regions, but not the union itself. So the operation of taking the convex hull has strictly increased the capacity region for this multiple access channel.

7. Convexity of capacity region of multiple access channel. Let \( C \subseteq \mathbb{R}^2 \) be the capacity region of all achievable rate pairs \( R = (R_1, R_2) \) for the multiple access channel. Show that \( C \) is a convex set by using a timesharing argument.

Specifically, show that if \( R^{(1)} \) and \( R^{(2)} \) are achievable, then \( \lambda R^{(1)} + (1 - \lambda) R^{(2)} \) is achievable for \( 0 \leq \lambda \leq 1 \).

Solution: Convexity of Capacity Regions.

Let \( R^{(1)} \) and \( R^{(2)} \) be two achievable rate pairs. Then there exist a sequence of \((2^n R_1^{(1)}, 2^n R_2^{(1)}, n)\) codes and a sequence of \((2^n R_1^{(2)}, 2^n R_2^{(2)}, n)\) codes for the channel with \( P_e^n(1) \to 0 \) and \( P_e^n(2) \to 0 \). We will now construct a code of rate \( \lambda R_1^{(1)} + (1 - \lambda) R_2^{(2)} \).

For a code length \( n \), use the concatenation of the codebook of length \( \lambda n \) and rate \( R_1^{(1)} \) and the code of length \( (1 - \lambda)n \) and rate \( R_2^{(2)} \). The new codebook consists of all pairs of codewords and hence the number of \( X_1 \) codewords is \( 2^{\lambda n} R_1^{(1)} \cdot 2^{(1 - \lambda)n} R_2^{(2)} \), and hence the rate is \( \lambda R_1^{(1)} + (1 - \lambda) R_2^{(2)} \). Similarly the rate of the \( X_2 \) codeword is \( \lambda R_2^{(1)} + (1 - \lambda) R_2^{(2)} \).

We will now show that the probability of error for this sequence of codes goes to zero. The decoding rule for the concatenated code is just the combination of the decoding rule for the parts of the code. Hence the probability of error for the combined codeword is less than the sum of the probabilities for each part. For the combined code,

\[
P_e^n \leq P_e^{(\lambda n)}(1) + P_e^{(1 - \lambda)n}(2)
\]

(14.75)