Problem 5.11

For the noiseless case, the received signal $r(t) = s(t), 0 \leq t \leq T$.

(a) The correlator output is

$$y(T) = \int_{0}^{T} r(\tau) s(\tau) d\tau$$

$$y(T) = \int_{0}^{T} s^{2}(\tau) d\tau$$

$$= \int_{0}^{T} \sin^{2} \left( \frac{8\pi\tau}{T} \right) d\tau$$

$$= \int_{0}^{T} \frac{1}{2} \left[ 1 - \cos \left( \frac{16\pi\tau}{T} \right) \right] d\tau$$

$$= T/2$$

(b) The matched filter is defined by the impulse response

$$h(t) = s(T - t)$$

The matched filter output is therefore

$$y(t) = \int_{-\infty}^{\infty} r(\lambda) h(t - \lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} s(\lambda) s(T - t + \lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} \sin \left( \frac{8\pi\lambda}{T} \right) \sin \left( \frac{8\pi(T - t + \lambda)}{T} \right) d\lambda$$
= \frac{1}{2} \int_{-\infty}^{\infty} \cos \left[ \frac{8\pi(T-t)}{T} \right] d\lambda - \frac{1}{2} \int_{-\infty}^{\infty} \cos \left[ \frac{8\pi(T-t+\lambda)}{T} \right] d\lambda

Since \(-T < \lambda \leq 0\), we have

\[ y(t) = \frac{1}{2} \cos \left( \frac{8\pi(t-T)}{T} \right) \bigg|_{\lambda = 0}^{\lambda = -T} \]

\[ -\frac{1}{2} \cdot \frac{T}{8\pi} \sin \left( \frac{8\pi(T-t+2\lambda)}{T} \right) \bigg|_{\lambda = -T}^{\lambda = 0} \]

\[ = \frac{T}{2} \cos \left( \frac{8\pi(t-T)}{T} \right) - \frac{T}{16\pi} \sin \left( \frac{8\pi(t-T)}{T} \right) - \frac{T}{16\pi} \sin \left( \frac{8\pi(t+T)}{T} \right) \]

(c) When the matched filter output is sampled at \( t = T \), we get

\[ y(T) = T/2 \]

which is exactly the same as the correlator output determined in part (a).

**Problem 5.12**

(a) The matched filter for signal \( s_1(t) \) is defined by the impulse response

\[ h_1(t) = s_1(T-t) \]

The matched filter for signal \( s_2(t) \) is defined by the impulse response

\[ h_2(t) = s_2(T-t) \]

The matched filter receiver is as follows

![Diagram of matched filter receiver](Image)
The distance between the two signal points $s_1(t)$ and $s_2(t)$ is

$$d = \sqrt{2E} = \sqrt{6T}$$

The average probability of error is therefore

$$P_e = \frac{1}{2} \text{erfc} \left( \frac{1}{2} \frac{d}{\sqrt{N_0}} \right)$$

$$= \frac{1}{2} \text{erfc} \left( \frac{1}{2} \frac{\sqrt{2E}}{\sqrt{N_0}} \right)$$

For $E/N_0$, we therefore have

$$P_e = \frac{1}{2} \text{erfc} \left( \frac{1}{2} \sqrt{2 \times 4} \right)$$

$$= \frac{1}{2} \text{erfc} (\sqrt{2})$$

$$= 4 \times 10^{-2}$$

Problem 5.13

Energy of binary symbol 1 represented by signal $s_1(t)$ is

$$E_1 = \int_0^{T/2} (+1)^2 \, dt + \int_{T/2}^T (-1)^2 \, dt = T$$

Energy of binary symbol 0 represented by signal $s_2(t)$ is the same as shown by
\[ E_2 = \int_0^{T/2} (-1)^2 \, dt + \int_{T/2}^{T} ( + 1)^2 \, dt = T \]

The only basis function of the signal-space diagram is

\[ \phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}} \]

The signal-space diagram of the Manchester code using the doublet pulse is as follows:

\[ \cdot \cdot \cdot \bullet 0 \bullet \sqrt{T} \]

Hence, the distance between the two signal points is \( d = 2\sqrt{T} \). The average probability of error over an AWGN channel is given by

\[ P_e = \frac{1}{2} \text{erfc} \left( \frac{d}{2 \sqrt{N_0}} \right) = \frac{1}{2} \text{erfc} \left( \frac{T}{\sqrt{N_0}} \right) \]

**Problem 5.14**

(a) Let \( Z \) denote the total observation space, which is divided into two parts \( Z_0 \) and \( Z_1 \). Whenever an observation falls in \( Z_0 \), we say \( H_0 \), and whenever an observation falls in \( Z_1 \), we say \( H_1 \). Thus, expressing the risk \( R \) in terms of the conditional probability density functions and the decision regions, we may write

\[ R = \sum_{0}^{0} P_0 \int_{Z_0} f_{X|H_0}(x|H_0) \, dx \]
Problem 5.17

(a) The minimum distance between any two adjacent signal points in the constellation of Fig. P5.17a of the textbook is

\[ d_{\text{min}}^{(a)} = 2\alpha \]

The minimum distance between any two adjacent signal points in the constellation of Fig. P5.17b of the textbook is

\[ d_{\text{min}}^{(b)} = \sqrt{(\sqrt{2}\alpha)^2 + (\sqrt{2}\alpha)^2} = 2\alpha \]

which is the same as \( d_{\text{min}}^{(a)} \). Hence, the average probability of symbol error using the constellation of Fig. P5.17a is the same as that of Fig. P5.17b.

(b) The constellation of Fig. P5.17a has minimum energy, whereas that of Fig. P5.17b is of non-minimum energy. Applying the minimum energy translate to the constellation of Fig. P5.17b, which involves translating it bodily to the left along the \( \phi_1 \)-axis by the amount \( \sqrt{2}\alpha \), we get the corresponding minimum energy configuration:

![Diagram](image)

Problem 5.18

Consider a set of three orthogonal signals denoted by \( \{s_i(t)\}_{i=0}^{2} \), each with energy \( E_s \). The average of these three signals is

\[ a(t) = \frac{1}{3} \sum_{i=0}^{2} s_i(t) \]

Applying the minimum energy translate to the signal set \( \{s_i(t)\}_{i=0}^{2} \), we get a new signal set defined by

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\[ s'_1(t) = -\frac{\sqrt{E}}{2} \phi_0(t) + \frac{\sqrt{3E}}{2} \phi_1(t) \] (4)

The remaining signal \( s'_2(t) \) may be expressed in terms of \( \phi_0(t) \) and \( \phi_1(t) \) as

\[ s'_2(t) = -\frac{\sqrt{E}}{2} \phi_0(t) - \frac{\sqrt{3E}}{2} \phi_1(t) \] (5)

Thus, using Eqs. (3) to (5), we may represent the simplex code by the following signal-space diagram:

\[ \text{Problem 5.19} \]

(a) An upper bound on the complementary error function is given by

\[ \text{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi} u} \]

Hence, we may bound the given \( P_e \) as follows:

\[ P_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) < \frac{\exp \left( \frac{-E_b}{N_0} \right)}{2 \sqrt{\pi E_b N_0}} = \frac{1}{2} \exp \left( \frac{-E_b}{N_0} \right) \times \sqrt{\frac{N_0}{\pi E_b}} \] (1)

For large positive \( u \), we may further simplify the upper bound on the complementary error function as shown here:
erfc(\(u\)) < \frac{\exp(-u^2)}{\sqrt{\pi}}

Correspondingly, we may bound \(P_e\) as follows:

\(P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi}}\)  

(b) For \(E_b/N_0 = 9\), we get the following results:

(i) The exact calculation of \(P_e\) yields

\[P_e = \frac{1}{2} \text{erfc}(3)\]

\[= 1.0 \times 10^{-5}\]

(ii) Using the bound in (1), we have the approximate value:

\[P_e \approx \frac{\exp(-9)}{6\sqrt{\pi}}\]

\[= 1.16 \times 10^{-5}\]

(iii) Using the looser bound of (2), we have

\[P_e \approx \frac{\exp(-9)}{2\sqrt{\pi}}\]

\[= 3.48 \times 10^{-5}\]

As expected, the first bound is more accurate than the second bound for calculating \(P_e\).

Problem 5.20

According to Eq. (5.91) of the textbook, the probability of error is over-bounded as follows: