Problem 7.1.1

random variable that indicates the result of flip 33. The PMF of $X_{33}$ is

$$P_{X_{33}}(x) = \begin{cases} 
1 - p & x = 0 \\
p & x = 1 \\
0 & \text{otherwise}
\end{cases}$$

Note that each $X_i$ has expected value $E[X] = p$ and variance $\text{Var}[X] = p(1 - p)$. The random variable $Y = X_1 + \cdots + X_{100}$ is the number of heads in 100 coin flips. Hence, $Y$ has the binomial PMF

$$P_Y(y) = \begin{cases} 
\binom{100}{y} p^y (1-p)^{100-y} & y = 0, 1, \ldots, 100 \\
0 & \text{otherwise}
\end{cases}$$

Since the $X_i$ are independent, by Theorems 7.1 and 7.3, the mean and variance of $Y$ are

$$E[Y] = 100E[X] = 100p \quad \text{Var}[Y] = 100 \text{Var}[X] = 100p(1 - p)$$

Problem 7.1.2

(a) Since $Y = X_1 + (-X_2)$, Theorem 7.1 says that the expected value of the difference is


(b) By Theorem 7.2, the variance of the difference is

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = 2 \text{Var}[X]$$

Problem 7.2.2

$$f_{X,Y}(x,y) = \begin{cases} 
1 & 0 \leq x, y \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

Proceeding as in Problem 7.2.1, we must first find $F_W(w)$ by integrating over the square defined by $0 \leq x, y \leq 1$. Again we are forced to find $F_W(w)$ in parts as we did in Problem 7.2.1 resulting in the following integrals for their appropriate regions. For $0 \leq w \leq 1$,

$$F_W(w) = \int_0^w \int_0^{w-x} dx dy = w^2/2$$

For $1 \leq w \leq 2$,

$$F_W(w) = \int_0^{w-1} \int_0^1 dx dy + \int_1^w \int_0^{w-y} dx dy = 2w - 1 - w^2/2$$
The complete expression for the CDF of $W$ is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \leq w \leq 1 \\ 2w - w^2/2 & 1 \leq w \leq 2 \\ 1 & \text{otherwise} \end{cases}$$

With the CDF, we can find $f_W(w)$ by differentiating with respect to $w$.

$$f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2 - w & 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 7.3.1**

For a constant $a > 0$, a zero mean Laplace random variable $X$ has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \quad -\infty < x < \infty$$

The moment generating function of $X$ is

$$\phi_X(s) = E[e^{sx}] = \frac{a}{2} \int_{-\infty}^{0} e^{sx} e^{ax} \, dx + \frac{a}{2} \int_{0}^{\infty} e^{sx} e^{-ax} \, dx$$

$$= \frac{a}{2} e^{(s+a)x} \bigg|_{0}^{\infty} + \frac{a}{2} e^{(s-a)x} \bigg|_{0}^{\infty}$$

$$= \frac{a}{2} \left( \frac{1}{s+a} - \frac{1}{s-a} \right)$$

$$= \frac{a^2}{a^2 - s^2}$$

**Problem 7.4.2**

points $X_i$ that you earn for game $i$ has PMF

$$P_{X_i}(x) = \begin{cases} 1/3 & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) The MGF of $X_i$ is

$$\phi_{X_i}(s) = E[e^{sX_i}] = 1/3 + e^s/3 + e^{2s}/3$$

Since $Y = X_1 + \cdots + X_n$, Theorem 7.10 implies

$$\phi_Y(s) = [\phi_{X_i}(s)]^n = [1 + e^s + e^{2s}]^n / 3^n$$

(b) First we observe that first and second moments of $X_i$ are

$$E[X_i] = \sum_x xP_{X_i}(x) = 1/3 + 2/3 = 1$$

$$E[X_i^2] = \sum_x x^2P_{X_i}(x) = 1^2/3 + 2^2/3 = 5/3$$
Hence, \( \text{Var}[X_i] = E[X_i^2] - (E[X_i])^2 = 2/3 \). By Theorems 7.1 and 7.3, the mean and variance of \( Y \) are

\[
E[Y] = nE[X] = n \\
\text{Var}[Y] = n \text{Var}[X] = 2n/3
\]

**Problem 7.5.2**

Using the moment generating function of \( X \), \( \phi_X(s) = e^{\sigma^2 s^2/2} \). We can find the \( n \)th moment of \( X \), \( E[X^n] \) by taking the \( n \)th derivative of \( \phi_X(s) \) and setting \( s = 0 \).

\[
E[X] = \sigma^2 s e^{\sigma^2 s^2/2} \bigg|_{s=0} = 0 \\
E[X^2] = \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2} \bigg|_{s=0} = \sigma^2
\]

Continuing in this manner we find that

\[
E[X^3] = (3\sigma^4 s + \sigma^6 s^3) e^{\sigma^2 s^2/2} \bigg|_{s=0} = 0 \\
E[X^4] = (3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4) e^{\sigma^2 s^2/2} \bigg|_{s=0} = 3\sigma^4
\]