Problem 4.7.7

Since the microphone voltage $V$ is uniformly distributed between -1 and 1 volts, $V$ has PDF and CDF:

$$f_V(v) = \begin{cases} 1/2 & -1 \leq v \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad F_V(v) = \begin{cases} 0 & v < -1 \\ (v + 1)/2 & -1 \leq v \leq 1 \\ 1 & v > 1 \end{cases}$$

The voltage is processed by a limiter whose output magnitude is given by:

$$L = \begin{cases} |V| & |V| \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

(a) 

$$P[L = 0.5] = P[|V| \geq 0.5] = P[V \geq 0.5] + P[V \leq -0.5]$$

$$= 1 - F_V(0.5) + F_V(-0.5)$$

$$= 1 - 1.5/2 + 0.5/2 = 1/2$$

(b) For $0 \leq l \leq 0.5$,

$$F_L(l) = P[|V| \leq l] = P[-l \leq v \leq l] = F_V(l) - F_V(-l) = 1/2(l + 1) - 1/2(-l + 1) = l$$

So the CDF of $L$ is:

$$F_L(l) = \begin{cases} 0 & l < 0 \\ l & 0 \leq l < 0.5 \\ 1 & l \geq 0.5 \end{cases}$$

(c) By taking the derivative of $F_L(l)$, the PDF of $L$ is:

$$f_L(l) = \begin{cases} 1 + (0.5)\delta(l - 0.5) & 0 \leq l \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of $L$ is:

$$E[L] = \int_{-\infty}^{\infty} l f_L(l) \, dl = \int_{0}^{0.5} l \, dl + 0.5 \int_{0}^{0.5} l(0.5)\delta(l - 0.5) \, dl = 0.375$$
Problem 4.7.13
shown in the following figure:

(a) Note that $Y = 1/2$ if and only if $0 \leq X \leq 1$. Thus,

$$P[Y = 1/2] = P[0 \leq X \leq 1] = \int_0^1 f_X(x) \, dx = \int_0^1 (x/2) \, dx = 1/4$$

(b) Since $Y \geq 1/2$, we can conclude that $F_Y(y) = 0$ for $y < 1/2$. Also, $F_Y(1/2) = P[Y = 1/2] = 1/4$. Similarly, for $1/2 < y \leq 1$,

$$F_Y(y) = P[0 \leq X \leq 1] = P[Y = 1/2] = 1/4$$

Next, for $1 < y \leq 2$,

$$F_Y(y) = P[X \leq y] = \int_0^y f_X(x) \, dx = y^2/4$$

Lastly, since $Y \leq 2$, $F_Y(y) = 1$ for $y \geq 2$. The complete expression of the CDF is

$$F_Y(y) = \begin{cases} 
0 & y < 1/2 \\
1/4 & 1/2 \leq y \leq 1 \\
y^2/4 & 1 < y < 2 \\
1 & y \geq 2 
\end{cases}$$

Problem 4.8.1

$$f_X(x) = \begin{cases} 
1/10 & -5 \leq x \leq 5 \\
0 & \text{otherwise} 
\end{cases}$$

(a) The event $B$ has probability

$$P[B] = P[-3 \leq X \leq 3] = \int_{-3}^{3} \frac{1}{10} \, dx = \frac{3}{5}$$

From Definition 4.15, the conditional PDF of $X$ given $B$ is

$$f_{X|B}(x) = \begin{cases} 
f_X(x) / P[B] & x \in B \\
0 & \text{otherwise} 
\end{cases}$$

(b) Given $B$, we see that $X$ has a uniform PDF over $[a, b]$ with $a = -3$ and $b = 3$. From Theorem 4.7, the conditional expected value of $X$ is $E[X|B] = (a + b)/2 = 0$.

(c) From Theorem 4.7, the conditional variance of $X$ is $\text{Var}[X|B] = (b - a)^2/12 = 3$. 

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Problem 4.8.2

\( Y \) is

\[
fy(y) = \begin{cases} 
(1/5) e^{-y/5} & y \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

(a) The event \( A \) has probability

\[
P[A] = P[Y < 2] = \int_{0}^{2} (1/5) e^{-y/5} dy = -e^{-y/5} \bigg|_{0}^{2} = 1 - e^{-2/5}
\]

From Definition 4.15, the conditional PDF of \( Y \) given \( A \) is

\[
f_{Y|A}(y) = \begin{cases} 
f_Y(y) / P[A] & x \in A \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
(1/5) e^{-y/5} / (1 - e^{-2/5}) & 0 \leq y < 2 \\
0 & \text{otherwise}
\end{cases}
\]

(b) The conditional expected value of \( Y \) given \( A \) is

\[
E[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y) dy = \frac{1/5}{1 - e^{-2/5}} \int_{0}^{2} ye^{-y/5} dy
\]

Using the integration by parts formula \( \int u dv = uv - \int v du \) with \( u = y \) and \( dv = e^{-y/5} dy \) yields

\[
E[Y|A] = \frac{1/5}{1 - e^{-2/5}} \left( -5ye^{-y/5} \bigg|_{0}^{2} + \int_{0}^{2} 5e^{-y/5} dy \right)
\]

\[
= \frac{1/5}{1 - e^{-2/5}} \left( -10e^{-2/5} - 25e^{-2/5} \bigg|_{0}^{2} \right)
\]

\[
= \frac{5 - 7e^{-2/5}}{1 - e^{-2/5}}
\]

Problem 5.1.2

(a) Because the probability that any random variable is less than \(-\infty\) is zero, we have

\[
F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0
\]

(b) The probability that any random variable is less than infinity is always one.

\[
F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x)
\]

(c) Although \( P[Y \leq \infty] = 1 \), \( P[X \leq -\infty] = 0 \). Therefore the following is true.

\[
F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0
\]

(d) Part (d) follows the same logic as that of part (a).

\[
F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0
\]

(e) Analogous to Part (b), we find that

\[
F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y)
\]
Problem 5.2.1

(a) The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

To find the constant $c$ we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c \, dy \, dx = cx - \frac{cx^2}{2} \bigg|_0^1 = \frac{c}{2} = 1$$

Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph

$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dy \, dx$$

$$= \int_0^{1/2} (2 - 4x) \, dx$$

$$= 1/2$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure at right. Here we can set up the following integrals

$$P[X + Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2 \, dy \, dx$$

$$= \int_0^{1/2} (1 - 2x) \, dx$$

$$= 1/2 - 1/4 = 1/4$$

Problem 5.2.2

Given the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
(a) To find the constant $c$ integrate $f_{X,Y}(x,y)$ over the all possible values of $X$ and $Y$ to get

$$1 = \int_0^1 \int_0^1 cxy^2 \, dx \, dy = c/6$$

Therefore $c = 6$.

(b) The probability $P[X \geq Y]$ is the integral of the joint PDF $f_{X,Y}(x,y)$ over the indicated shaded region.

$$P[X \geq Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx$$
$$= \int_0^1 2x^4 \, dx$$
$$= 2/5$$

Similarly, to find $P[Y \leq X^2]$ we can integrate over the region shown in the figure.

$$P[Y \leq X^2] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx = 1/4$$

(c) Here we can choose to either integrate $f_{X,Y}(x,y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing

$$P[\min(X,Y) \leq 1/2] = 1 - P[\min(X,Y) > 1/2]$$
$$= 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2 \, dx \, dy$$
$$= 1 - \int_{1/2}^1 9y^2/4 \, dy = 11/32$$

(d) The $P[\max(X,Y) \leq 3/4]$ can be found be integrating over the shaded region shown below.
\[ P[\max(X,Y) \leq 3/4] = P[X \leq 3/4, Y \leq 3/4] \]
\[ = \int_0^{3/4} \int_0^{3/4} 6xy^2 \, dx \, dy \]
\[ = \left( x^2 \left|_{0}^{3/4} \right. \right) \left( y^3 \left|_{0}^{3/4} \right. \right) \]
\[ = (3/4)^5 = 0.237 \]

**Problem 5.3.1**

(a) The joint PDF (and the corresponding region of nonzero probability) are

\[ f_{X,Y}(x,y) = \begin{cases} 
1/2 & -1 \leq x \leq y \leq 1 \\
0 & \text{otherwise} 
\end{cases} \]

(b)

\[ P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2} \, dy \, dx = \int_0^1 \frac{1-x}{2} \, dx = 1/4 \]

This result can be deduced by geometry. The shaded triangle of the \(X,Y\) plane corresponding to the event \(X > 0\) is 1/4 of the total shaded area.

(c) For \(x > 1\) or \(x < -1\), \(f_X(x) = 0\). For \(-1 \leq x \leq 1\),

\[ f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_x^1 \frac{1}{2} \, dy = (1-x)/2 \]

The complete expression for the marginal PDF is

\[ f_X(x) = \begin{cases} 
(1-x)/2 & -1 \leq x \leq 1 \\
0 & \text{otherwise} 
\end{cases} \]

(d) From the marginal PDF \(f_X(x)\), the expected value of \(X\) is

\[ E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \frac{1}{2} \int_{-1}^{1} x(1-x) \, dx = \frac{x^2}{4} - \frac{x^3}{6} \bigg|_{-1}^{1} = -\frac{1}{3} \]
Problem 5.3.2

\[ f_{X,Y}(x,y) = \begin{cases} 
 2 & x + y \leq 1, x, y \geq 0 \\
 0 & \text{otherwise}
\end{cases} \]

Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions.

\[ f_X(x) = \int_0^{1-x} 2dy = \begin{cases} 
 2(1-x) & 0 \leq x \leq 1 \\
 0 & \text{otherwise}
\end{cases} \]

Likewise for \( f_Y(y) \):

\[ f_Y(y) = \int_0^{1-y} 2dx = \begin{cases} 
 2(1-y) & 0 \leq y \leq 1 \\
 0 & \text{otherwise}
\end{cases} \]

Problem 5.4.1

(a) The minimum value of \( W \) is \( W = 0 \), which occurs when \( X = 0 \) and \( Y = 0 \). The maximum value of \( W \) is \( W = 1 \), which occurs when \( X = 1 \) or \( Y = 1 \). The range of \( W \) is \( S_W = \{w|0 \leq w \leq 1\} \).

(b) For \( 0 \leq w \leq 1 \), the CDF of \( W \) is

\[ F_W(w) = P[\max(X,Y) \leq w] = P[X \leq w, Y \leq w] = \int_0^w \int_0^w f_{X,Y}(x,y) \, dy \, dx \]

Substituting \( f_{X,Y}(x,y) = x + y \) yields

\[ F_W(w) = \int_0^w \int_0^w (x+y) \, dy \, dx = \int_0^w \left( x + \frac{y^2}{2} \right)_{y=0}^{y=w} \, dx = \int_0^w (wx + w^2/2) \, dx = w^3 \]

The complete expression for the CDF is

\[ F_W(w) = \begin{cases} 
 0 & w < 0 \\
 w^3 & 0 \leq w \leq 1 \\
 1 & \text{otherwise}
\end{cases} \]

(c) The PDF of \( W \) is found by differentiating the CDF:

\[ f_Y(y) = \frac{dF_W(w)}{dw} = \begin{cases} 
 3w^2 & 0 \leq w \leq 1 \\
 0 & \text{otherwise}
\end{cases} \]
Problem 5.4.2

(a) Since the joint PDF \( f_{X,Y}(x,y) \) is nonzero only for \( 0 \leq y \leq x \leq 1 \), we observe that \( W = Y - X \leq 0 \) since \( Y \leq X \). In addition, the most negative value of \( W \) occurs when \( Y = 0 \) and \( X = 1 \) and \( W = -1 \). Hence the range of \( W \) is \( S_W = \{w \mid -1 \leq w \leq 0\} \).

(b) For \( w < -1 \), \( F_W(w) = 0 \). For \( w > 0 \), \( F_W(w) = 1 \). For \(-1 \leq w \leq 0 \), the CDF of \( W \) is

\[
F_W(w) = P[Y - X \leq w] = \int_{-w}^{1} \int_{0}^{x+w} 6y \, dy \, dx
= \int_{-w}^{1} 3(x+w)^2 \, dx = (x+w)^3 |_{-w}^{1} = (1+w)^3
\]

Therefore, the complete CDF of \( W \) is

\[
F_W(w) = \begin{cases} 
0 & w < -1 \\
(1+w)^3 & -1 \leq w \leq 0 \\
1 & w > 0
\end{cases}
\]

(c) By taking the derivative of \( f_W(w) \) with respect to \( w \), we obtain the PDF

\[
f_W(w) = \begin{cases} 
3(w+1)^2 & -1 \leq w \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]