Diversity and Multiplexing:
A Fundamental Tradeoff in Wireless Systems

David Tse
Department of EECS, U.C. Berkeley

April 14, 2003

DIMACS
Wireless Fading Channels

- Fundamental characteristic of wireless channels: multi-path fading.
Wireless Fading Channels

- Fundamental characteristic of wireless channels: multi-path fading.
- Two important resources of a fading channel: diversity and degrees of freedom.
A channel with more diversity has smaller probability in deep fades.
Example: Spatial Diversity

- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- Repeat and Average: compensate against channel unreliability.
Example: Spatial Diversity

- Additional independent fading channels increase diversity.
Example: Spatial Diversity

- Additional independent fading channels increase diversity.
- Spatial diversity
Example: Spatial Diversity

- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit
Example: Spatial Diversity

- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
• Additional independent fading channels increase diversity.
• Spatial diversity: receive, transmit or both.
• Repeat and Average: compensate against channel unreliability.
Signals arrive in multiple directions provide multiple degrees of freedom for communication. The same effect can be obtained via scattering even when antennas are close together.
Signals arrive in multiple directions provide multiple degrees of freedom for communication. Same effect can be obtained via scattering even when antennas are close together.
Signals arrive in multiple directions provide multiple degrees of freedom for communication.
Signals arrive in multiple directions provide multiple degrees of freedom for communication.
Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Same effect can be obtained via scattering even when antennas are close together.
The two resources have been considered mainly in isolation: existing schemes focus on maximizing either the diversity gain or the multiplexing gain.
Diversity vs. Multiplexing

The two resources have been considered mainly in isolation: existing schemes focus on maximizing either the diversity gain or the multiplexing gain.

The right way of looking at the problem is a tradeoff between the two types of gain.
The two resources have been considered mainly in isolation: existing schemes focus on maximizing either the diversity gain or the multiplexing gain.

The right way of looking at the problem is a tradeoff between the two types of gain.

The optimal tradeoff achievable by a coding scheme gives a fundamental performance limit on communication over fading channels.
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- cooperative relaying systems
\[ y_t = H_t x_t + w_t, \quad w_t \sim \mathcal{CN}(0, 1) \]

- Rayleigh flat fading i.i.d. across antenna pairs \((h_{ij} \sim \mathcal{CN}(0, 1))\).
- SNR is the average signal-to-noise ratio at each receive antenna.
Coherent Block Fading Model

- Focus on codes over $l$ symbols, where $H$ remains constant.
- $H$ is known to the receiver but not the transmitter.
- Assumption valid as long as

$$l \ll \text{coherence time \times coherence bandwidth}.$$
Space-Time Block Code

\[ Y = HX + W \]

Focus on coding over a single block of length \( l \).
Diversity Gain

Motivation: Binary Detection

\[ y = hx + w \quad P_e \approx P(\|h\| \text{ is small}) \propto \text{SNR}^{-1} \]

\[
\begin{align*}
  y_1 &= h_1 x + w_1 \\
  y_2 &= h_2 x + w_2
\end{align*}
\]

\[ P_e \approx P(\|h_1\|, \|h_2\| \text{ are both small}) \propto \text{SNR}^{-2} \]
Motivation: Binary Detection

\[ y = hx + w \quad \Rightarrow \quad P_e \approx P(\|h\| \text{ is small }) \propto \text{SNR}^{-1} \]

\[ y_1 = h_1x + w_1 \quad y_2 = h_2x + w_2 \quad \Rightarrow \quad P_e \approx P(\|h_1\|,\|h_2\| \text{ are both small}) \propto \text{SNR}^{-2} \]

General Definition

A space-time coding scheme achieves diversity gain \( d \), if

\[ P_e(\text{SNR}) \sim \text{SNR}^{-d} \]
Spatial Multiplexing Gain

Motivation: Channel capacity (Telatar '95, Foschini'96)

\[ C(\text{SNR}) \approx \min\{m, n\} \log \text{SNR}(\text{bps}/\text{Hz}) \]

\[ \min\{m, n\} \text{ degrees of freedom to communicate.} \]
Spatial Multiplexing Gain

Motivation: Channel capacity (Telatar’ 95, Foschini’96)

\[ C(\text{SNR}) \approx \min\{m, n\} \log \text{SNR}(\text{bps/Hz}) \]

\( \min\{m, n\} \) degrees of freedom to communicate.

Definition A space-time coding scheme achieves spatial multiplexing gain \( r \), if

\[ R(\text{SNR}) = r \log \text{SNR}(\text{bps/Hz}) \]
Fundamental Tradeoff

A space-time coding scheme achieves

Spatial Multiplexing Gain $r$ : $R = r \log \text{SNR}$ (bps/Hz)

and

Diversity Gain $d$ : $P_e \approx \text{SNR}^{-d}$
Fundamental Tradeoff

A space-time coding scheme achieves

Spatial Multiplexing Gain $r$ : $R = r \log \text{SNR}$ (bps/Hz)

and

Diversity Gain $d$ : $P_e \approx \text{SNR}^{-d}$

Fundamental tradeoff: for any $r$, the maximum diversity gain achievable: $d^*_{m,n}(r)$.

$r \rightarrow d^*_{m,n}(r)$
**Fundamental Tradeoff**

A space-time coding scheme achieves

Spatial Multiplexing Gain $r$ : $R = r \log \text{SNR}$ (bps/Hz)

and

Diversity Gain $d$ : $P_e \approx \text{SNR}^{-d}$

Fundamental tradeoff: for any $r$, the maximum diversity gain achievable: $d_{m,n}^*(r)$.

$r \rightarrow d_{m,n}^*(r)$

A tradeoff between data rate and error probability.
**Main Result: Optimal Tradeoff**

(Zheng and Tse 02)

$m$: # of Tx. Ant.

$n$: # of Rx. Ant.

$l$: block length

$l \geq m + n - 1$

$d$: diversity gain

$P_e \approx \text{SNR}^{-d}$

$r$: multiplexing gain

$R = r \log \text{SNR}$

For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

\( m \): # of Tx. Ant.
\( n \): # of Rx. Ant.
\( l \): block length
\( l \geq m + n - 1 \)

\( d \): diversity gain
\( P_e \approx \text{SNR}^{-d} \)

\( r \): multiplexing gain
\( R = r \log \text{SNR} \)

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d(r) \)

For integer \( r \), it is as though \( r \) transmit and \( r \) receive antennas were dedicated for multiplexing and the rest provide diversity.
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

\( m \): # of Tx. Ant.
\( n \): # of Rx. Ant.
\( l \): block length
\( l \geq m + n - 1 \)

\( d \): diversity gain
\( P_e \approx SNR^{-d} \)

\( r \): multiplexing gain
\( R = r \log SNR \)
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

\( m \): # of Tx. Ant.
\( n \): # of Rx. Ant.
\( l \): block length
\( l \geq m + n - 1 \)

\( d \): diversity gain
\( P_e \approx \text{SNR}^{-d} \)

\( r \): multiplexing gain
\( R = r \log \text{SNR} \)

For integer \( r \), it is as though \( r \) transmit and \( r \) receive antennas were dedicated for multiplexing and the rest provide diversity.
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

\( m \): # of Tx. Ant.
\( n \): # of Rx. Ant.
\( l \): block length
\( l \geq m + n - 1 \)

\( d \): diversity gain
\( P_e \approx \text{SNR}^{-d} \)

\( r \): multiplexing gain
\( R = r \log \text{SNR} \)

For integer \( r \), it is as though \( r \) transmit and \( r \) receive antennas were dedicated for multiplexing and the rest provide diversity.
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

\[ m: \text{\# of Tx. Ant.} \]
\[ n: \text{\# of Rx. Ant.} \]
\[ l: \text{block length} \]
\[ l \geq m + n - 1 \]

\[ d: \text{diversity gain} \]
\[ P_e \approx \text{SNR}^{-d} \]

\[ r: \text{multiplexing gain} \]
\[ R = r \log \text{SNR} \]

For integer \( r \), it is as though \( r \) transmit and \( r \) receive antennas were dedicated for multiplexing and the rest provide diversity.
What do I get by adding one more antenna at the transmitter and the receiver?
Adding More Antennas

\[ m: \text{# of Tx. Ant.} \]
\[ n: \text{# of Rx. Ant.} \]
\[ l: \text{block length} \]
\[ l \geq m + n - 1 \]

\[ d: \text{diversity gain} \]

\[ r: \text{multiplexing gain} \]

Spatial Multiplexing Gain: \[ r = \frac{R}{\log \text{SNR}} \]

Diversity Advantage: \[ d(r) \]

Capacity result: increasing \( \min\{m, n\} \) by 1 adds 1 more degree of freedom.

Tradeoff curve: increasing both \( m \) and \( n \) by 1 yields multiplexing gain +1 for any diversity requirement \( d \).
Adding More Antennas

$m$: # of Tx. Ant.
$n$: # of Rx. Ant.
$l$: block length
$l \geq m + n - 1$

d: diversity gain

$r$: multiplexing gain

- **Capacity result**: increasing $\min\{m, n\}$ by 1 adds 1 more degree of freedom.
Adding More Antennas

$m$: # of Tx. Ant.
$n$: # of Rx. Ant.
$l$: block length
$l \geq m + n - 1$

$d$: diversity gain

$r$: multiplexing gain

- **Capacity result**: increasing $\min\{m, n\}$ by 1 adds 1 more degree of freedom.
- **Tradeoff curve**: increasing both $m$ and $n$ by 1 yields multiplexing gain $+1$ for any diversity requirement $d$. 
Sketch of Proof

Lemma:
For block length $l \geq m + n - 1$, the error probability of the best code satisfies at high SNR:

$$P_e(SNR) \approx P(\text{Outage}) = P(I(H) < R)$$

where

$$I(H) = \log \det [I + \text{SNR}HH^*]$$

is the mutual information achieved by the i.i.d. Gaussian input.
Outage Analysis

\[ P(\text{Outage}) = P\{\log \det[I + \text{SNRHH}^\dagger] < R} \]

- In scalar $1 \times 1$ channel, outage occurs when the channel gain $\|h\|^2$ is small.
Outage Analysis

\[ P(\text{Outage}) = P\{\log \det[I + \text{SNR}{HH}^\dagger] < R \} \]

- In scalar \(1 \times 1\) channel, outage occurs when the channel gain \(\|h\|^2\) is small.
- In general \(m \times n\) channel, outage occurs when some or all of the singular values of \(H\) are small. There are many ways for this to happen.
Outage Analysis

\[ P(\text{Outage}) = P\{\log \det[I + \text{SNR}HH^\dagger] < R\} \]

- In scalar \(1 \times 1\) channel, outage occurs when the channel gain \(\|h\|^2\) is small.

- In general \(m \times n\) channel, outage occurs when some or all of the singular values of \(H\) are small. There are many ways for this to happen.

- Let \(v = \text{vector of singular values of } H\):
  
  Laplace Principle:

  \[ P(\text{Outage}) \approx \min_{v \in \text{Out}} \text{SNR}^{-f(v)} \]
Geometric Picture (integer $r$)

Scalar Channel

\[ \text{Result: at rate } R = r \log \text{SNR}, \text{ for } r \text{ integer, outage occurs typically when } H \text{ is in or close to the set } \{ H : \text{rank}(H) \leq r \} \], with \( \epsilon^2 = \text{SNR} - 1 \).

The dimension of the normal space to the sub-manifold of rank $r$ matrices within the set of all $M \times N$ matrices is \( (M-r)(N-r) \).

\[ P(\text{Outage}) \approx \text{SNR} - (M-r)(N-r) \].
Geometric Picture (integer $r$)

Scalar Channel

Result: At rate $R = r \log_{10} \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{ H : \text{rank}(H) \leq r \}$, with $\epsilon^2 = \text{SNR} - 1$.

The dimension of the normal space to the sub-manifold of rank $r$ matrices within the set of all $M \times N$ matrices is $(M - r)(N - r)$.

$P(\text{Outage}) \approx \text{SNR} - (M - r)(N - r)$
Geometric Picture (integer $r$)

Scalar Channel

Vector Channel

$\epsilon$

Bad $H$  Good $H$

$\text{All } n \times m \text{ Matrices}$

$\text{Rank}(H) = r$

Result:

At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{H: \text{rank}(H) \leq r\}$, with $\epsilon^2 = \text{SNR} - 1$.

The co-dimension of the manifold of rank $r$ matrices within the set of all $m \times n$ matrices is $(m - r)(n - r)$. 

$P(\text{Outage}) \approx \text{SNR} - (M - r)(N - r)$
Result: At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{H : \text{rank}(H) \leq r\}$,
**Geometric Picture (integer \( r \))**

Result: At rate \( R = r \log \text{SNR} \), for \( r \) integer, outage occurs typically when \( H \) is close to the set \( \{ H : \text{rank}(H) \leq r \} \), with \( \epsilon^2 = \text{SNR}^{-1} \).
Result: At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{H : \text{rank}(H) \leq r\}$, with $\epsilon^2 = \text{SNR}^{-1}$.

The co-dimension of the manifold of rank $r$ matrices within the set of all $m \times n$ matrices is $(m - r)(n - r)$.

$$P(\text{Outage}) \approx \text{SNR}^{-(m-r)(n-r)}$$
Piecewise Linearity of Tradeoff Curve

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d \times (r) \)

- \((0, mn)\)
- \((r, (m-r)(n-r))\)
- \((\min(m,n), 0)\)

Multiple Antenna
  \( m \times n \) channel

Single Antenna
  \( 1 \) channel

For non-integer \( r \), qualitatively same outage behavior as \( \lfloor r \rfloor \) but with larger \( \epsilon \).

Scalar channel: qualitatively same outage behavior for all \( r \).

Vector channel: qualitatively different outage behavior in different segments of the tradeoff curve.
Tradeoff Analysis of Specific Designs

Focus on two transmit antennas.

\[ Y = HX + W \]

Repetition Scheme:

\[
X = \begin{bmatrix}
  x_1 & 0 \\
  0 & x_1
\end{bmatrix}
\]

\[ y_1 = \|H\|x_1 + w_1 \]

Alamouti Scheme:

\[
X = \begin{bmatrix}
  x_1 & -x^*_2 \\
  x^*_2 & x_1
\end{bmatrix}
\]

\[ [y_1 y_2] = \|H\|[x_1 x_2] + [w_1 w_2] \]
Comparison: 2 × 1 System

Repetition: \( y_1 = \|H\|x_1 + w \)

Alamouti: \( [y_1 y_2] = \|H\|[x_1 x_2] + [w_1 w_2] \)
Comparison: $2 \times 1$ System

Repetition: $y_1 = \|H\| x_1 + w$

Alamouti: $[y_1 y_2] = \|H\| [x_1 x_2] + [w_1 w_2]$

- Spatial Multiplexing Gain: $r = R / \log \text{SNR}$
- Diversity Gain: $d^*(r) = (1/2,0), (0,2), (1,0)$
- Alamouti Repetition
Comparison: 2 × 1 System

Repetition: \( y_1 = \| H \| x_1 + w \)

Alamouti: \( [y_1 y_2] = \| H \| [x_1 x_2] + [w_1 w_2] \)

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d\ast (r) \)

Optimal Tradeoff
Comparison: 2 × 2 System

Repetition: \( y_1 = \|H\| x_1 + w \)

Alamouti: \( [y_1 \ y_2] = \|H\| [x_1 \ x_2] + [w_1 \ w_2] \)

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d^*(r) \)

(1/2, 0)

(0, 4)

Diversity Gain: \( d^*(r) \)

Repetition

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)
Comparison: 2 × 2 System

Repetition: \( y_1 = \|H\| x_1 + w \)

Alamouti: \( [y_1 \ y_2] = \|H\| [x_1 \ x_2] + [w_1 \ w_2] \)

Spatial Multiplexing Gain: \( r = R/\log \text{SNR} \)

Diversity Gain: \( d^*(r) \)

(0,4) Alamouti

(1/2,0) Repetition

(1,0)
Comparison: 2 × 2 System

Repetition: \[ y_1 = \|H\| x_1 + w \]

Alamouti: \[ [y_1 y_2] = \|H\| [x_1 x_2] + [w_1 w_2] \]

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d^* \) (1/2, 0) (1, 0) (0, 4) (1, 1) (2, 0)

Optimal Tradeoff
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- cooperative relaying systems
In a point-to-point link, multiple antennas provide diversity and multiplexing gain.

In a system with $K$ users, multiple antennas can be used to discriminate signals from different users too.

Continue assuming i.i.d. Rayleigh fading, $n$ receive antennas, $m$ transmit antennas *per user.*
Suppose we want every user to achieve an error probability:

\[ P_e \sim \text{SNR}^{-d} \]

and a data rate

\[ R = r \log \text{SNR} \quad \text{bits/s/Hz.} \]

What is the optimal tradeoff between the diversity gain \( d \) and the multiplexing gain \( r \)?

Assume a coding block length \( l \geq Km + n - 1 \).
Optimal Multiuser D-M Tradeoff: \( m \leq n/(K + 1) \)

(Tse, Viswanath and Zheng 02)

In this regime, diversity-multiplexing tradeoff of each user is as though it is the only user in the system, i.e. \( d_{m,n}^*(r) \)
**Multiuser Tradeoff**: \( m > n/(K + 1) \)

Single-user diversity-multiplexing tradeoff up to \( r^* = n/(K + 1) \).
Multiuser Tradeoff: \( m > n/(K+1) \)

Single-user diversity-multiplexing tradeoff up to \( r^* = m/(K+1) \).

For \( r \) from \( n/(K+1) \) to \( \min\{n/K, m\} \), tradeoff is as though the \( K \) users are pooled together into a single user with \( Km \) antennas and rate \( Kr \), i.e. \( d^*_{K,m,n}(Kr) \).
Benefit of Dual Transmit Antennas

Question: what does adding one more antenna at each mobile buy me?

Assume there are more users than receive antennas.
Benefit of Dual Transmit Antennas

Question: what does adding one more antenna at each mobile buy me?
Assume there are more users than receive antennas.
Answer

Spatial Multiplexing Gain: \( r = \frac{R}{\log SNR} \)

Diversity Gain: \( d(r) \)

Optimal tradeoff

Adding one more transmit antenna does not increase the number of degrees of freedom for each user. However, it increases the maximum diversity gain from \( N \) to \( 2N \).

More generally, it improves the diversity gain \( d(r) \) for every \( r \).
Adding one more transmit antenna does not increase the number of degrees of freedom for each user.

However, it increases the maximum diversity gain from $n$ to $2n$.

More generally, it improves the diversity gain $d(r)$ for every $r$. 

\[ \text{Spatial Multiplexing Gain: } r = \frac{R}{\log \text{SNR}} \]

\[ \text{Diversity Gain: } d(r) \]
Consider only the case of $m = 1$ transmit antenna for each user and number of users $K < n$. 
Maximum diversity gain is $n - K + 1$: “costs $K - 1$ diversity gain to null out $K - 1$ interferers.” (Winters, Salz and Gitlin 93)
Maximum diversity gain is \( n - K + 1 \): “costs \( K - 1 \) diversity gain to null out \( K - 1 \) interferers.” (Winters, Salz and Gitlin 93)

Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.
Tradeoff for the Decorrelator

Maximum diversity gain is $n - K + 1$: “costs $K - 1$ diversity gain to null out $K - 1$ interferers.” (Winters, Salz and Gitlin 93)

Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.

Optimal tradeoff curve is also a straight line but with a maximum diversity gain of $n$.

Adding one receive antenna provides more reliability per user and accommodate 1 more user.
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- cooperative relaying systems
Cooperative relaying protocols can be designed via a
diversity-multiplexing tradeoff analysis. (Laneman, Tse, Wornell 01)
Cooperative relaying protocols can be designed via a diversity-multiplexing tradeoff analysis.

(Laneman, Tse and Wornell 01)
Tradeoff Curves of Relaying Strategies

- Multiplexing gain
- Diversity gain

Direct transmission

1/2 1 Multiplexing gain

2 1 Diversity gain
Cooperative Relaying

Tx 1
Tx 2
Rx
Channel 2
Channel 1
Cooperation
Tradeoff Curves of Relaying Strategies

![Diagram showing tradeoff curves for relaying strategies with axes labeled Diversity gain and Multiplexing gain, marked with points for direct transmission and a line connecting them.](image-url)
Tradeoff Curves of Relaying Strategies

- **Diversity gain**
- **Multiplexing gain**

- Direct transmission
- Amplify + forward

![Diagram showing tradeoff curves for relaying strategies with axes for diversity gain and multiplexing gain, indicating points for direct transmission and amplify + forward.]
Tradeoff Curves of Relaying Strategies

- Direct transmission
- Amplify and forward

Gain vs. Diversity

- Multiplexing gain
- Diversity gain

Graph:
- Direct transmission
- Amplify + forward

Values:
- $\frac{1}{2}$
- 1
- 2
Cooperative Relaying

Tx 1
Tx 2
Rx

Channel 1
Channel 2

Cooperation
Tradeoff Curves of Relaying Strategies

Multiplexing gain

Diversity gain

1 ½

1

2

amplify + forward

direct transmission

1 Multiplexing gain

½
Tradeoff Curves of Relaying Strategies

![Graph showing tradeoff curves for direct transmission, amplify + forward, and amplify + forward + ack strategies. The graph includes axes for direct transmission gain and diversity gain with corresponding lines for each strategy.]
Conclusion

Diversity-multiplexing tradeoff is a unified way to look at performance over wireless channels.

Future work:

- Code design.
- Application to other wireless scenarios.
- Extension to channel-uncertainty-limited rather than noise-limited regime.