1. **Eigenfunctions and Signal Space** Suppose a waveform $\phi_k(t)$ is applied to linear time-invariant filter with impulse response $h(t)$ and output $r(t)$.

(a) If $\phi_k(t)$ is zero outside $t \in [0, T]$ and we only consider the output on the same interval, find an expression which can be solved to find the set of all $\phi_k(t)$ such that

$$r(t) = (\phi_k * h)(t) = \lambda_k \phi_k(t)$$

where $\lambda_k \in C$.

Such $\phi_k(t)$ are called eigenfunctions of the system.

(b) For $h(t) = e^{-t}$ provide numerical examples of a few $\phi_k(t)$ and the associated $\lambda_k$ for $T = 1$. Plot the two with the largest eigenvalues and the two with the smallest eigenvalues. You may numerically calculate any integrals.

(c) Suppose $\phi_k(t)$ and $r(t)$ are now defined $\forall t$. Show (or disprove) that $e^{st}$ is an eigenfunction of any $h(t)$ where $s \in C$.

(d) Now suppose we require the $\phi_k(t)$ to be orthonormal. Then we cannot guarantee that $\phi_k(t)$ is an eigenfunction of the system $h(t)$. However, if we define $\psi_k(t)$ as the output associated with application of $\phi_k(t)$ to $h(t)$, then we can require

$$\langle \phi_k(t), \phi_\ell(t) \rangle = \delta_{k\ell}$$

and

$$\langle \psi_k(t), \psi_\ell(t) \rangle = \lambda_k \delta_{k\ell}$$

Derive an expression from which the $\phi_k(t)$ and thence the $\psi_k(t)$ can be found. Assume the $\phi_k()$ and $\psi_k()$ are defined only on $(0, T)$.

(e) Let $h(t) = e^{-t} u(t)$. Derive an expression for the $\phi_k(t)$, find a few of them numerically. Plot the two with the largest eigenvalues and the two with the smallest eigenvalues. Assume $T = 1$.

(f) Let $h(t) = \frac{\sin 2\pi t}{\pi}$ and repeat the previous part for $T = 1$ and then $T = 100$.

(g) Let $\phi_k(t)$ and $\psi_k(t)$ now be defined $\forall t$. Analytically find appropriate $\phi_k(t)$ for arbitrary $h(t)$. 


2. **Waterfilling Math** You are given a power budget $P$ which you can split over $N$ Gaussian channels. Let the noise level in each channel be $\sigma_i$ and the power used on that channel $p_i$. The capacity of each channel is

$$C_i = \frac{1}{2} \log \left( 1 + \frac{p_i}{\sigma_i} \right)$$

and the total capacity using all channels is

$$C = \sum_{i=1}^{N} C_i = \frac{1}{2} \sum_{i=1}^{N} \log \left( 1 + \frac{p_i}{\sigma_i} \right)$$

(a) Suppose you are only allowed to place power into a single dimension, but you can choose which one. Show that any choice of dimension $k$ must satisfy $\sigma_k \leq \sigma_\ell$, $\ell = 1, 2, \cdots, N$ to attain maximum capacity $C_k$.

(b) Prove using Lagrange Multipliers that

$$p_i = [c - \sigma_i]^+$$

where $c$ is a constant chosen to satisfy $\sum_i p_i = P$.

Derive a condition on $P$ so that no $p_i$ is zero.

You can find writeups on Lagrange Multipliers in most optimization texts and some applied math texts. My favorite is Advanced Calculus for Applications (Hildebrand).

(c) Let $d_i = p_i + \sigma_i$. Show using Jensen’s inequality that to maximize capacity the $d_i$ should be a constant $c$, or if not, $\sigma_i$. What is the constant $c$? Is this the same result as the previous part?

**HINT:** This is a little of a mind-bender. For those of you who’ve done a formal optimization course, this is an example of optimization through a “slack variable” (I think). Once you get the basic idea, it might help when finishing up to recognize that $\log 1 = 0$ and $\log d_i \geq 0$ by the definition of $d_i$ and capacity.

3. **Single Base Interference Avoidance** In class and in the assigned Interference Avoidance paper (see web page) we learned the following facts:

$$S = \begin{bmatrix} | & | & | & | \\ s_1 & s_2 & \cdots & s_M \end{bmatrix}$$

where the $\{s_k\}$ are codewords of dimension $N$. We usually assume $M \geq N$ but that’s written in stone. We also have

$$SS^T = SS^T$$

$$R = SS^T + W$$

$$C_s = \frac{1}{2} \log |R| + \frac{1}{2} \log |W|$$

Greedy interference avoidance applied to codeword $k$ replaces $s_k$ with $x$ where $x$ is a minimum eigenvalue eigenvector of $R - s_k s_k^T$ and we derived a bunch of conditions for fixed
points and optimality in class and in the paper, especially in relation to a quantity called the TSC (or GSC), defined as

\[ \text{TSC} = \text{Trace} \left( \left( SS^T \right)^2 \right) \]

\[ \text{GSC} = \text{Trace} \left( R^2 \right) \]

(a) We know that in white noise, greedy interference avoidance does not increase TSC. Show that greedy interference avoidance does not increase GSC in colored noise (\( W \neq \alpha I \)).

(b) Show that greedy interference avoidance does not decrease sum capacity.

(c) Write an interference avoidance algorithm (in Matlab or C). You may assume white unit variance noise \( W = I \). Starting from randomly chosen unit norm codewords, sketch (plot) the codeword set attained by interference avoidance for \( M = 3 \) users in \( N = 2 \) dimensions. Do the same for \( M = 8 \) users in \( N = 2 \) dimensions. Can you find examples of initial codeword sets (for \( M = 3 \)) which do not attain minimum TSC even after repeated application of your algorithm? Plot the codeword covariance spectra in each case.

4. **Multi-Base Interference Avoidance** Now, suppose we have a TWO base system and two user groups with codeword matrices \( S_1 \) and \( S_2 \). The users are arranged so that the covariance at base 1 is

\[ R_1 = S_1 S_1^T + g S_2 S_2^T + I \]

and

\[ R_2 = S_2 S_2^T + g S_1 S_1^T + I \]

at base 2 where \( g \) is a non-negative gain factor representing the interference level seen at base \( i \) from user set \( j \) (a symmetric interference system). Assume \( N = 3 \) and \( M = 3 \) for each set of users (for a total of six users over the two bases).

Write code which applies greedy interference avoidance at each base. For \( g = 0.1 \) and \( g = 2 \) first try sequential application (one base then the other). Then try interleaved application and answer the following questions.

(a) One could define two TSC’s for this problem:

\[ \text{TSC}_1 = \text{Trace} \left( \left( S_1 S_1^T \right)^2 \right) \]

and

\[ \text{TSC}_2 = \text{Trace} \left( \left( S_2 S_2^T \right)^2 \right) \]

Plot each TSC on the same graph as a function of algorithm step. When (if ever) do either or both of the TSC’s converge? Is there a pattern of convergence as a function of the interference gain factor \( g \)?

(b) If the codewords converge, plot the converged spectra for both user sets. Do the codeword spectra overlap in signal space? If so, when and how much?

5. **Potential Research Paper Topics** COMING!