Problem Solution: Yates and Goodman, 3.5.3

**Problem 3.5.3**

First we observe that for $n = 1, 2, \ldots$, the marginal PMF of $N$ satisfies

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = (1 - p)^{n-1} p \sum_{k=1}^{n} \frac{1}{n} = (1 - p)^{n-1} p$$

Thus, the event $B$ has probability

$$P[B] = \sum_{n=10}^{\infty} P_N(n) = (1 - p)^9 [1 + (1 - p) + (1 - p)^2 + \cdots] = (1 - p)^9$$

From Theorem 3.11,

$$P_{N,K|B}(n,k) = \begin{cases} \frac{P_{N,K}(n,k)}{P[B]} & n, k \in B \\ 0 & \text{otherwise} \end{cases}$$

However, we can also find it just by summing the conditional joint PMF.

$$P_{N|B}(n) = \sum_{k=1}^{n} P_{N,K|B}(n,k) = \begin{cases} (1 - p)^{n-10} p/n & n = 10, 11, \ldots; k = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

The conditional PMF $P_{N|B}(n|b)$ could be found directly from $P_N(n)$ using Theorem 2.19. Instead, however, we observe that given $B$, $N' = N - 9$ has a geometric PMF with mean $1/p$. That is, for $n = 1, 2, \ldots$,

$$P_{N'|B}(n) = P[N = n + 9|B] = P_{N|B}(n + 9) = (1 - p)^{n-1} p$$

Hence, given $B$, $N = N' + 9$ and we can calculate the conditional expectations


$$\text{Var}[N|B] = \text{Var}[N'|B] = (1 - p)/p^2$$

Note that further along in the problem we will need $E[N^2|B]$ which we now calculate.

$$E[N^2|B] = \text{Var}[N|B] + (E[N|B])^2$$

$$= \frac{2}{p^2} + \frac{17}{p} + 81$$
For the conditional moments of $K$, we work directly with the conditional PMF $P_{N,K|B}(n,k)$.

$$E[K|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} \frac{k(1-p)^{n-10}p}{n} = \sum_{n=10}^{\infty} \frac{(1-p)^{n-10}p}{n} \sum_{k=1}^{n} k$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[K|B] = \sum_{n=10}^{\infty} \frac{n+1}{2} (1-p)^{n-10}p = \frac{1}{2} E[N+1|B] = \frac{1}{2}p + 5$$

We now can calculate the conditional expectation of the sum.

$$E[N+K|B] = E[N|B] + E[K|B] = \frac{1}{p} + 9 + 1/(2p) + 5 = \frac{3}{2p} + 14$$

The conditional second moment of $K$ is

$$E[K^2|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} k^2 \frac{(1-p)^{n-10}p}{n} = \sum_{n=10}^{\infty} \frac{(1-p)^{n-10}p}{n} \sum_{k=1}^{n} k^2$$

Using the identity $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, we obtain

$$E[K^2|B] = \sum_{n=10}^{\infty} \frac{(n+1)(2n+1)}{6} (1-p)^{n-10}p = \frac{1}{6} E[(N+1)(2N+1)|B]$$

Applying the values of $E[N|B]$ and $E[N^2|B]$ found above, we find that

$$E[K^2|B] = \frac{E[N^2|B]}{3} + \frac{E[N|B]}{2} + \frac{1}{6} = \frac{2}{3p^2} + \frac{37}{6p} + 31 \frac{2}{3}$$

Thus, we can calculate the conditional variance of $K$.

$$\text{Var}[K|B] = E[K^2|B] - (E[K|B])^2 = \frac{5}{12p^2} - \frac{7}{6p} + 6 \frac{2}{3}$$

To find the conditional correlation of $N$ and $K$,

$$E[NK|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} nk \frac{(1-p)^{n-10}p}{n} = \sum_{n=10}^{\infty} \frac{(1-p)^{n-10}p}{n} \sum_{k=1}^{n} k$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[NK|B] = \sum_{n=10}^{\infty} \frac{n(n+1)}{2} (1-p)^{n-10}p = \frac{1}{2} E[N(N+1)|B] = \frac{1}{p^2} + \frac{9}{p} + 45$$