Analysis of
Proportional Fair Sharing Algorithms
Harold J. Kushner and Philip Whiting

$N$ users, each with infinite backlog.

Time is divided into small scheduling intervals.

In each interval, one of the $N$ users is chosen to transmit.

Rates of transmission for each user are randomly time varying.

If user $i$ is selected in interval $n$, then it transmits $r_{i,n}$ units of data. Known at start of interval.

$\{r_{i,n}, n < \infty \}$ bounded, correlated, random sequence, which can be correlated among the $i$. Need not be stationary.

More complex models later.

Define throughput up to time $n$ for user $i$:

$$\theta_{i,n} = \frac{1}{n} \sum_{l=1}^{n} r_{i,l} I_{i,l},$$

where $I_{i,l} = 1$ if user $i$ is chosen and is zero otherwise:

$$\theta_{i,n+1} = \theta_{i,n} + \epsilon_n [I_{i,n+1} r_{i,n+1} - \theta_{i,n}] = \theta_{i,n} + \epsilon_n Y_{i,n},$$

where $\epsilon_n = 1/(n + 1)$. 
Discounted throughput.

We also use the discounted form, for $0 < \epsilon < 1$:

$$\theta_{i,n}^\epsilon = (1 - \epsilon)\epsilon^n \theta_{i,0}^\epsilon + (1 - \epsilon) \sum_{l=1}^{n} \epsilon^{n-l} r_{i,l} I_{i,l}$$

which can be written recursively as

$$\theta_{i,n+1}^\epsilon = (1 - \epsilon)\theta_{i,n}^\epsilon + \epsilon \left[ I_{i,n+1} r_{i,n+1} - \theta_{i,n}^\epsilon \right].$$

$\epsilon$ is chosen to balance the needs of estimating throughput (requiring a small value) with the ability to track changes in the channel characteristics (requiring a larger value).

$\epsilon$ should be chosen small enough so that it provides an acceptable measure of the throughput.

We are concerned with the asymptotic properties of $\theta_n$ and $\theta_n^\epsilon$. The limit is unique, irrespective of the starting value, and has certain optimality properties.
The PFS Rule

For small $d_i > 0$, in interval $n + 1$ use the standard rule (balance between rate and record)

$$\arg \max \left\{ \frac{r_{i,n+1}}{d_i + \theta_{i,n}}, i \leq N \right\}.$$ 

In the event of ties, randomize among the possibilities. The end results are independent of how the conflicts are resolved.

Define the vectors $\theta_n = \{\theta_{i,n}, i \leq N\}$ and $R_n = \{r_{i,n}, i \leq N\}$. Define the shifted process $\theta^n(\cdot)$ (with components $\theta^n_i(\cdot)$) by:

Define $\theta^e(\cdot)$ analogously: $\theta^e(t) = \theta^n_{e}, \quad t \in [n\epsilon, n\epsilon + \epsilon)$. 

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A SA Theorem

**Theorem 1.** (Kushner, Yin [Theorems 2.2 and 2.3, Section 8.2].) Conditions to be listed later. The limit (w.p 1 or weak sense) paths $\theta^n(\cdot)$ satisfy the ODE

$$\dot{\theta}_i = E r_i I_{\{r_i/(\theta_i+d_i) \geq r_j/(\theta_j+d_j), j \neq i\}} - \theta_i \equiv \bar{h}_i(\theta) - \theta_i.$$  

(Note that $\bar{g}_i(\theta)$ is the “mean” rate for user $i$ at parameter $\theta$.) The same conclusion holds if the $\epsilon_n = 1/(n+1)$ is replaced by $\epsilon_n \to 0$, $\Sigma_n \epsilon_n = \infty$, and where for some sequence $\alpha_n \to \infty$

$$\lim_{n} \sup_{0 \leq l \leq \alpha_n} \left| \frac{\epsilon_{n+l}}{\epsilon_n} - 1 \right| = 0.$$ 

The limit points of $\theta_n$ are those of the ODE.

For the discounted algorithm, the same conclusion holds (weak convergence) for $\theta^\epsilon(t_\epsilon + \cdot)$ for any $t_\epsilon$ sequence. If $t_\epsilon \to \infty$, then the limit points are those of the ODE.

If the $\{R_n\}$ are identically distributed and have a density $p(\cdot)$, then the choice is unique with probability one for each $\theta \in \mathbb{R}^N_0$ and $\bar{h}_i(\cdot)$ is differentiable (and continuously differentiable, if $p(\cdot)$ is continuous).

**Theorem 2.** There is a unique limit point, $\bar{\theta}$, irrespective of the initial condition. Hence the processes $\theta^n(\cdot), (n \to \infty)$ and $\theta^\epsilon(t_\epsilon + \cdot)$ (for $t_\epsilon \to \infty$, $\epsilon \to 0$) converge to this value.
Assumptions: 1

A3.0. Let $E_n$ denote conditioning on the past: $\xi_n = \{R_i : l \leq n\}$. For each $i, n, \xi_n$, 
$$h_{i,n}(\theta, \xi_n) = E_r_i,n+1 I_{\{r_i,n+1/(d_i+\theta_i) \geq r_j,n+1/(d_j+\theta_j), j \neq i\}}$$
is continuous in $\theta$. Let $\delta > 0$ be arbitrary. Then in the set \{\theta : \theta_i \geq \delta, i \leq N\}, the continuity is uniform in $n$ and in $\xi_n$.

A3.1a. \{R_n, n < \infty\} is stationary. Define $\bar{h}_i(\cdot)$ by the stationary expectation:
$$\bar{h}_i(\theta) = E_r_i I_{\{r_i/(d_i+\theta_i) \geq r_j/(d_j+\theta_j), j \neq i\}}, \quad i \leq N.$$In (3.1), $\theta$ is considered fixed. Also,
$$\lim_{m,n \to \infty} \frac{1}{m} \sum_{i=n}^{n+m-1} \left[ E_r_i,n+1 I_{\{r_i,n+1/(d_i+\theta_i) \geq r_j,n+1/(d_j+\theta_j), j \neq i\}} - \bar{h}_i(\theta) \right] = 0$$
in the sense of probability. There are small positive $\delta$ and $\delta_1$ such that
$$P \{r_i,n/d_i \geq r_j,n/(d_j - \delta) + \delta_1, j \neq i\} > 0, \quad i \leq N.$$

A3.1b. (Not needed for Theorem 1.) $R_n$ is defined on some bounded set and has a bounded density.
On the smoothness of $\bar{h}(\cdot)$. By (A3.1b), $\bar{h}(\cdot)$ is Lipschitz continuous.

To see this, consider the two dimensional case and let $p(\cdot)$ denote the density of $R_n$.

Let $\theta \in \mathbb{R}_+^N$ and write $w = (d_1 + \theta_1)/(d_2 + \theta_2)$. Then

$$\bar{h}_1(\theta) = \int r_1 I_{\{r_2 \geq w\}}p(r_1, r_2)dr_1dr_2,$$

which is Lipschitz continuous with respect to $w$, since the area of the region where the indicator is not zero is a differentiable function of $w$.

The derivative of $\bar{h}_i(\cdot)$ will be continuous if $p(\cdot)$ is bounded and continuous.

More general conditions later.
Comments on the assumptions.

The last part of (A3.1a) is innocuous and is used to assure that when a component $\theta_i$ is very small there is a nonzero chance that user $i$ will be chosen, no matter what the values of the other components of $\theta$. It guarantees that $\bar{h}_i(\theta)$ is positive when $\theta_i$ is small.

The density assumption (A3.1b) is satisfied under standard physical assumptions. Indeed all the assumptions hold under Raleigh fading if the channels are independent. It is used only to show that the limit point is unique.

Condition (*) is a very weak form of the law of large numbers, due to the use of the conditional expectation $E_n$. If the conditional expectation of the transmitted rate at time $l$, given the data to time $n$, is close to its stationary expectation for large positive $l - n$, then it holds. If the channel rate process is ergodic, then the condition holds even without the conditional expectation.

Condition (A3.0) asks that slight changes in $\theta_n$ would change the conditional (on data to the present) expectation of the next accepted rates only slightly.
A Variation: Discretized or quantized rates,

Suppose that \( r_{i,n} \) are the theoretical rates in that there is in principle a transmission scheme that could realize them, perhaps by adjusting the symbol interval and coding in each scheduling interval. In applications, it might be possible to transmit at only one of a discrete set of rates.

The algorithms and results are readily adjusted to accommodate this need.

Continue to make the assignments using the values of the \( r_{i,n+1}, i \leq N \). But use the true transmitted rate, called \( r_{n,i}^d \) in computing the throughput:

\[
\theta_{i,n+1}^f = \theta_{i,n}^f + \epsilon \left( I_{i,n+1}^f r_{i,n+1}^d - \theta_{i,n}^f \right).
\]

The results are similar.
A two-user example.

Two independent users with received signal power determined by stationary Rayleigh fading and with constant external noise.

Rates are proportional to the SNR, with mean rates $1/\beta_i$. Then

$$\dot{\theta}_1 = \frac{1}{\beta_1} - \frac{\beta_1(d_1 + \theta_1)^2}{(\beta_1(d_1 + \theta_1) + \beta_2(d_2 + \theta_2))^2} - \theta_1$$

$$\dot{\theta}_2 = \frac{1}{\beta_2} - \frac{\beta_2(d_2 + \theta_2)^2}{(\beta_1(d_1 + \theta_1) + \beta_2(d_2 + \theta_2))^2} - \theta_2$$

For $d_i \equiv 0$, $\bar{\theta}_i = 3/4 \cdot 1/\beta_i$
An interpretation of PFS: maximizing a utility function.

Define the utility function

\[ U(\theta) = \sum_i \log(d_i + \theta_i). \]

An alternative view of the assignment rule is that it maximizes
\[ U(\theta_{n+1}) - U(\theta_n) \]
to first order in the \( \epsilon_n \), as seen by the following argument.

By a first order Taylor expansion,

\[ U(\theta_{n+1}) - U(\theta_n) = \epsilon_n \sum_i \frac{r_{i,n+1}I_{i,n+1} - \theta_{i,n}}{d_i + \theta_{i,n}} + O(\epsilon_n^2). \]

Since

\[ \sum_i I_{i,n+1} \equiv 1, \]

the argmax rule maximizes the first order term.

The rule maximizes \( \lim_n U(\theta_n) \) over all other admissible rules.

**Theorem 3.** There is no assignment policy which yields a limit throughput \( \tilde{\theta} \neq \bar{\theta} \) such that \( U(\tilde{\theta}) \geq U(\bar{\theta}) \).
\[
\begin{align*}
\dot{\theta}_1 &= -\theta_1 + r_1 \\
\dot{\theta}_2 &= -\theta_2 \\
\end{align*}
\]

\[
\begin{align*}
\dot{\theta}_1 &= -\theta_1 \\
\dot{\theta}_2 &= -\theta_2 + r_2 \\
\end{align*}
\]

\(r_1, r_2\) constant. Discontinuous dynamics.
Dynamical Systems Concepts

For vectors $X, Y \in \mathbb{R}^N$, we write $X \geq Y$ (resp., $X > Y$) if $X_i \geq Y_i$ for all $i$ (resp., and, in addition $X \neq Y$).

If $X_i > Y_i$ for all $i$, then we write $X \gg Y$.

Consider

$$\dot{x} = f(x), \; x \in \mathbb{R}^N, \quad f(\cdot) \text{ Lipschitz continuous.}$$

Write $x(t|y)$ for the solution when the initial condition is $y$.

The function $f(\cdot)$ is said to satisfy the Kamke if for any $x, y$ and $i$, satisfying $x \leq y$ and $x_i = y_i$, we have $f_i(x) \leq f_i(y)$.

In our case, $f(\theta) = \bar{h}(\theta) - \theta$, and the condition holds.

The system is said to be cooperative.

The Kamke condition implies the following monotonicity result.

**Theorem 4.** [? , Proposition 1.1] Let $f(\cdot)$ be Lipschitz continuous and assume the Kamke condition. If $x(0) \leq y(0)$ (resp., $<, \ll$, then $x(t|x(0)) \leq x(t|y(0)))$ (resp., $<, \ll$).
Proofs

Theorem 1 is a standard SA result. See the book of Kushner and Yin.

Proof of Theorem 2. We need only prove the uniqueness of the limit point of the ODE.

The path $\theta(t|\theta(0))$ is bounded, uniformly in the $\theta(0)$ in any compact set, and all paths tend to some compact set as $t \to \infty$.

Also (say, by shifting the time origin), we can suppose that there is $\delta > 0$ such that $\theta_i(t|\theta(0)) \geq \delta$ for all $t$.

Since the path is initially monotone increasing in each coordinate when started near the origin (since $\tilde{h}(\theta) - \theta \gg 0$ for $\theta \approx 0$), the monotonicity (Theorem 4), implies that it will be monotonic (non-decreasing) in each coordinate for all $t$, for any initial condition sufficiently close to the origin.

Thus, there is a unique limit point for the path $\theta(t|\theta(0))$ for each $\theta(0)$ near the origin.

Let $\tilde{\theta}$ and $\tilde{\theta}$ be limit points, corresponding to initial conditions $\tilde{\theta}(0)$ and $\tilde{\theta}(0)$, resp., arbitrarily close to the origin.

Since both paths are monotonically increasing near the origin, without loss of generality, we can suppose that for some small $t_0 > 0$, $\theta(t_0|\tilde{\theta}(0)) \gg \tilde{\theta}(0)$.

Then, by letting $\theta(t_0|\tilde{\theta}(0))$ be the initial condition for a new path, we can suppose that $\theta(t|\tilde{\theta}(0)) \geq \theta(t|\tilde{\theta}(0))$ for all $t$.

Thus, $\tilde{\theta} \geq \tilde{\theta}$. 

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An analogous argument yields that $\bar{\theta} \geq \hat{\theta}$.

Thus there is a unique limit, $\bar{\theta}$, for all paths starting sufficiently close to the origin.

$\bar{\theta}$ is an equilibrium point; i.e., $h(\bar{\theta}) = \bar{\theta}$.

Furthermore, $\theta(t|\theta(0)) \leq \bar{\theta}$ for all $\theta(0)$ sufficiently close to the origin.

Now, consider the path starting at an arbitrary initial condition $\hat{\theta} \leq \bar{\theta}$. After some small time $t_0 > 0$, all components of the path will be positive and $\theta(t_0|\hat{\theta}) \geq \theta(0)$ for some $\theta(0)$ arbitrarily close to the origin.

Then, the monotonicity argument (and slightly shifting the time origin of one of the paths) yields $\theta(t|\hat{\theta}) \geq \theta(t|\theta(0))$ for all $t$.

Hence any limit point of $\theta(t|\hat{\theta})$ must be no smaller than $\bar{\theta}$.

But, by the monotonicity again, $\theta(t|\hat{\theta}) \leq \theta(t|\bar{\theta}) = \bar{\theta}$ for all $t$.

We can conclude that $\theta(t|\hat{\theta}) \to \bar{\theta}$ as $t \to \infty$. 

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Define $Q(\theta) = \{x : x \geq \theta\}$ and consider an arbitrary $\theta(0)$.

The monotonicity argument can be used again to show that all limit points are in $Q(\bar{\theta})$.

It only remains to show that any path starting in $Q(\bar{\theta})$ goes to $\bar{\theta}$.

So far, we have used only the monotonicity property and not any other properties of the stochastic approximation process that led to the ODE.

The rest of the details involve the properties of the argmax rule and essentially standard stochastic approximation arguments.
Suppose that there is \( \tilde{\theta} \in Q(\bar{\theta}), \tilde{\theta} \neq \bar{\theta} \):

\[
\dot{U}(\tilde{\theta}) = \sum_i (\tilde{h}_i(\tilde{\theta}) - \tilde{\theta}_i)/(d_i + \tilde{\theta}_i) \geq 0.
\]

(*)

Then, since \( \tilde{\theta} \geq \bar{\theta} \) and \( \tilde{\theta} \neq \bar{\theta} \),

\[
\sum_i (\tilde{h}_i(\tilde{\theta}) - \tilde{\theta}_i)/(d_i + \tilde{\theta}_i) > 0.
\]

(**)

Now, consider the slot allocation rule

\[
\arg \max_{i \leq N} \{ r_{i,n+1}/(d_i + \tilde{\theta}_i) \}.
\]

Let \( \tilde{I}_{i,n+1}^\epsilon \) = indicator function of the event that user \( i \) is chosen.

Modulo a second order error \( O(\epsilon^2)t \), the maximizing property of \( I_{n+1}^\epsilon \) yields

\[
U(\theta^\epsilon(t)) - U(\bar{\theta}) = \epsilon \sum_i \sum_{l=0}^{t/\epsilon-1} r^\epsilon_{i,l+1} I_{i,l+1}^\epsilon - \theta^\epsilon_{i,l} \geq \epsilon \sum_i \sum_{l=0}^{t/\epsilon-1} \tilde{r}_{i,l+1} I_{i,l+1}^\epsilon - \theta^\epsilon_{i,l}
\]

where \( \theta^\epsilon_i \) (and \( \theta^\epsilon(\cdot) \)) are the solutions under the original argmax rule.
\[ U(\theta^\varepsilon(t)) - U(\bar{\theta}) = \varepsilon \sum_i \sum_{l=0}^{t/\varepsilon-1} \frac{r_{i,l+1} I_{i,l+1}^\varepsilon - \theta_{i,l}^\varepsilon}{d_i + \theta_{i,l}^\varepsilon} \geq \varepsilon \sum_i \sum_{l=0}^{t/\varepsilon-1} \frac{r_{i,l+1} \tilde{I}_{i,l+1}^\varepsilon - \theta_{i,l}^\varepsilon}{d_i + \theta_{i,l}^\varepsilon} \]

The stochastic approximation arguments that led to Theorem 1, together with (**), imply that as \( \varepsilon \to 0 \) the limit \( \theta(\cdot) \) satisfies

\[ U(\theta(t)) - U(\bar{\theta}) = \int_0^t \sum_i \frac{\bar{h}_i(\theta(s)) - \theta_i(s)}{d_i + \theta_i(s)} ds \geq \int_0^t \sum_i \frac{\bar{h}_i(\bar{\theta}) - \theta_i(s)}{d_i + \theta_i(s)} ds. \]

This, together with the inequality in (***) implies that

\[ \dot{U}(\theta(t))|_{t=0} = \sum_i \frac{\bar{h}_i(\bar{\theta}) - \bar{\theta}_i}{d_i + \bar{\theta}_i} \geq \sum_i \frac{\bar{h}_i(\bar{\theta}) - \bar{\theta}_i}{d_i + \bar{\theta}_i} > 0. \]

But the first sum is zero since \( \bar{h}(\bar{\theta}) = \bar{\theta} \).

Thus, we have a contradiction to (*) and can conclude that \( \dot{U}(\theta) < 0 \) for all \( \theta \in Q(\bar{\theta}) - \bar{\theta} \).

This implies that \( \dot{U}(\theta(\cdot|\bar{\theta})) \) is strictly decreasing when the path is in \( Q(\bar{\theta}) - \bar{\theta} \), which implies that any path starting at some \( \theta(0) \in Q(\bar{\theta}) \) must end up at \( \bar{\theta} \).

Thus, \( \bar{\theta} \) is the unique limit point of the ODE, irrespective of the initial condition. Hence it is asymptotically stable.

End of proof.
Other strictly concave utility functions.

Consider the utility

\[ U(\theta) = \sum_i c_i / (\theta_i + d_i)^\gamma, \quad 0 < \gamma < 1, \; c_i > 0, \; d_i > 0. \]

Then

\[ \dot{U}(\theta) = \gamma \sum_i \frac{c_i \dot{\theta}_i}{(\theta_i + d_i)^{1-\gamma}}. \]

Thus the chosen user is

\[ \arg \max_{i \leq N} \left\{ \frac{c_i r_{i,n+1}}{(\theta_{i,n} + d_i)^{1-\gamma}} \right\}. \]

The ODE is

\[ \dot{\theta}_i = E r_i I_{\{c_ir_i/c_jr_j \geq [(\theta_i + d_i) / (\theta_j + d_j)]^{1-\gamma}, j \neq i\}} - \theta_i \]

This rule is not as sensitive to large values of \( \theta_i \) as is the original.

The analogs of Theorems 1–3 hold.

Thus, there is a wide choice of useful and convergent algorithms which allow a variety of tradeoffs between the current rates and throughputs in making the assignment.
Multiple Channels and/or Antennas.

Up to now, there was only a single channel. There are similar algorithms and results when there are multiple resources to be assigned.

Consider the following form: two transmitters, possibly different locations and frequencies, at same base station.

The associated channels will usually have different characteristics. Base the assignment on the log utility and the discounted throughput.

The simplest method of assignment is to assign each channel independently. Then one or both might be assigned to any user.

Equivalently, assign to maximize the first order term in \( U(\theta_{n+1}^e) - U(\theta_n^e) \). The assumptions are just those of Theorems 1–3, applied to each channel separately.

There is a unique globally asymptotically stable limit point \( \bar{\theta} \), and the argmax assignment algorithm maximizes the long run utility. The basic properties that were used in the proofs still hold. The ODE still satisfies the Kamke condition, and the argmax rule maximizes the increment in the utility to first order.

Let \( r_{i,j,n} \) =rates for user \( i \) in the channel from transmitter \( j \).

For the two user case,\[
\theta_{1,n+1}^e = (1 - \epsilon) \theta_{1,n}^e + \epsilon r_{11,n+1} I\{r_{11,n+1}/r_{21,n+1} \geq (d_1 + \theta_{1,n}^e)/(d_2 + \theta_{2,n}^e)\} \\
+ \epsilon r_{12,n+1} I\{r_{12,n+1}/r_{22,n+1} \geq (d_1 + \theta_{1,n}^e)/(d_2 + \theta_{2,n}^e)\},
\]
with the analogous formula for the other user.
Multiple resources, continued: Coordination

There was no coordination between the assignments of the two transmitters.

An alternative for more efficient use of the resource.

Keep the above choices. But also allow the possibility of the two antennas being used in a coordinated way for the same user, with (for example) space-time coding.

This simply adds another possible rate to be considered when using the argmax rule.

Space-time coding is selected only because it is one way of using both channels for the same user. The choice is still made for each scheduling interval, and might differ from interval to interval. The full assignment algorithm increases the channel capacity over what space-time coding used by itself could achieve.

One looks for the maximum of

\[
\frac{r_{11,n+1}}{\theta^c_{1,n} + d_1} + \frac{r_{22,n+1}}{\theta^c_{2,n} + d_2}, \quad \frac{r_{12,n+1}}{\theta^c_{1,n} + d_1} + \frac{r_{21,n+1}}{\theta^c_{2,n} + d_2}, \quad \frac{r^c_{1,n+1}}{\theta^c_{1,n} + d_1}, \quad \frac{r^c_{2,n+1}}{\theta^c_{2,n} + d_2}.
\]

Under natural analogs of (A3.0) and (A3.1a), Theorems 1–3 hold.

Thus, even for this more complicated case, there is a unique limit point and the algorithm is a utility maximizer.
<table>
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<th>-9.5</th>
<th>-8.5</th>
<th>-6.5</th>
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Table 1: Rate vs. SNR for 1% packet loss (taken from [?])

Figure 1: Time dependent behavior of Proportional Fair

**Advantages of PFS.**

Assume Raleigh fading and use Table 1. Figure 1 gives the behavior for three mobiles with mean SNRs, -12dB, -2dB, -8dB. We take $\epsilon = 0.0001$. The value of $\epsilon$ is determined by a balance between what is considered a reasonable measure of discounted throughput and the desire to track changing conditions. If there are 1000 slots per unit time, then (roughly speaking) $\epsilon = 0.0001$ corresponds to a measure over about 10 seconds.

There are two sets of curves: those with solid lines and (the higher ones) with dotted lines. For the solid lines, the SNRs (and hence the rates) are assumed to be constant at the average values.

The second set of curves are obtained for Rayleigh fading with fading rate 6 Hz and the same mean SNRs. The true current rates are used. The significant gains in the throughput for all mobiles are evident.
Figure 2: Sample path for $\theta$ and the Solution to the ODE

**Convergence to the ODE path.**

Consider two users with received signal power determined by a stationary Rayleigh fading and with constant external noise, Rates proportional to SNR, with mean rates $1/\beta_1, 1/\beta_2$. A Rayleigh fading simulator with $1/\beta_1 = 572$ bits/slot and $1/\beta_2 = 128$ bits/slot was used. The fading rates were taken as 60 Hz. In equilibrium the throughputs are, $\frac{3}{2} \cdot \frac{1}{2} \cdot 572 = 429, \frac{3}{2} \cdot \frac{1}{2} \cdot 128 = 96$. 
Iterate Averaging

Suppose that we wish to get a good estimate of $\bar{\theta}$ as quickly

$$\theta_{i,n+1} = \theta_{i,n} + \epsilon_n Y_{i,n}(\theta_n)$$

Define the Jacobian matrix $A = \bar{g}_\theta(\bar{\theta}) - I$ and

$$\Sigma_0 = \sum_{l=-\infty}^{\infty} EY_l(\bar{\theta})Y'_l(\bar{\theta}).$$

Consider the algorithm where $K$ is a matrix which we can choose but such that the iterations still converge to $\bar{\theta}$:

$$\theta_{n+1} = \theta_n + \frac{1}{n+1}KY_n.$$ 

Under appropriate conditions $(\theta_n - \bar{\theta})\sqrt{n}$ converges in distribution to a normally distributed random variable with mean zero and covariance

$$\int_0^\infty e^{(KA+I/2)t}K\Sigma_0K'e^{(A'K'+I/2)t}dt. \quad (*)$$

The optimal value of $K$ (in the sense of minimizing the trace of the covariance) is $K = -A^{-1}$, which is unknown.

Now, let $\epsilon_n \to 0$ slowly enough so that $n\epsilon_n \to \infty$.

Define $\Theta_n = \sum_{l=1}^n \theta_l/n$.

It turns out that $(\Theta_n - \bar{\theta})\sqrt{n}$ is also asymptotically normal with mean zero, and covariance $(*)$, but with the optimal $K$ used.