Principles of Communications Systems  

Spring 2005

There are THREE questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. (50 points) **Pulse Amplitude Modulation:** Consider the signal

\[ r(t) = m(t) \sum_{k=-\infty}^{\infty} p(t-kT) \]

where \( m(t) \) is program material bandlimited to \( \pm W \) Hz and \( p(t) \) is an arbitrary waveform such that the sum \( \sum_{k=-\infty}^{\infty} p(t-kT) \) exists.

(a) (20 points) Suppose \( p(t) = \delta(t) \). What condition on \( T \) insures that \( m(t) \) can always be recovered from \( r(t) \)?

**SOLUTION:** This is the form used to derive the Nyquist sampling theorem. We must have \( T < 1/2W \).

(b) (20 points) Now suppose \( W = 10, \ T = 10^{-3} \), and \( p(t) = \frac{\sin(\pi t)}{\pi t} \). Since \( p(t) \) exists for all time, you’ll notice that the pulses \( p(t) \) which comprise \( \sum_{k=-\infty}^{\infty} p(t-kT) \) OVERLAP. If we apply an ideal band pass filter \( H(f) = u(f+1010) - u(f+990)u(f-990) - u(f-1010) \) (where \( u() \) is the unit step function) to \( r(t) \), show how can \( m(t) \) be recovered from \( r(t) \) (or not).

**SOLUTION:** From class we know that the pulse sum, since it’s periodic, is going to be a set of impulses in frequency separated by \( 1/T \) and scaled by the spectrum of \( p(t) \). This can be easily derived by noting \( \sum_{k=-\infty}^{\infty} p(t-kT) = p(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT) \) We know that the Fourier transform of \( \sum_{k=-\infty}^{\infty} \delta(t-kT) \) is \( \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f-k/T) \). Convolution in time domain implies multiplication in frequency domain so \( \mathcal{F} \left[ p(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT) \right] = P(f) \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f-k/T) \)

However, the spectrum of \( p(t) \) (a sinc pulse) in this case is nonzero only for \( |f| \leq 1 \). So, the spectrum of \( r(t) \) is a scaled replica of \( M(f) \). Since \( m(t) \) is band limited to \( \pm 10 \) Hz, the output of the bandpass filter is identically zero and \( m(t) \) cannot be recovered.

(c) (20 points) Now suppose \( W = 10, \ T = 10^{-3} \), and \( p(t) = \frac{\sin(10^5 \pi t)}{10^5 \pi t} \). Since \( p(t) \) exists for all time, you’ll notice that the pulses \( p(t) \) which comprise \( \sum_{k=-\infty}^{\infty} p(t-kT) \) OVERLAP. If we apply an ideal band pass filter \( H(f) = u(f+1010) - u(f+990)u(f-990) - u(f-1010) \) (where \( u() \) is the unit step function) to \( r(t) \), show how can \( m(t) \) be recovered from \( r(t) \) (or not).

**SOLUTION:** From class we know that the pulse sum, since it’s periodic, is going to be a set of impulses in frequency separated by \( 1/T \) and scaled by the spectrum of \( p(t) \).
However, the spectrum of \( p(t) \) (a sinc pulse) in this case is nonzero only for \(|f| \leq 10^5\). So, the spectrum of \( r(t) \) is a bunch of scaled shifted replicas of \( M(f) \) each centered at \( f = 10^3 k \) where \( k \) is an integer. Since \( m(t) \) is band limited to \( \pm 10 \) Hz, the output of the bandpass filter is proportional to \( M(f + 10^3) + M(f - 10^3) \) and we can use an envelope detector (or synchronous AM demodulation to recover \( m(t) \)).

2. (50 points) Quantization:

(a) (20 points) What is the purpose of a quantizer? State your answer in words (no more than a short paragraph). NOTE: this is not an optimality question, just a simple question about what a quantizer is used for.

**SOLUTION:** The purpose of a quantizer is to approximate samples (usually of a waveform) using a finite set of amplitude levels. Such quantization is a precursor for digital transmission of a signal since samples of a continuous real-valued waveform cannot otherwise be represented with a finite number of bits.

(b) (30 points) We have seen in class that an optimal quantizer function \( Q(x) \) seems to always be non-decreasing. Please PROVE that this observation is true (or not). You may assume that the PDF of \( X \), the random variable to be quantized, exists and is non-zero for all \( X \in \mathbb{R} \). HINT: start from the Loyd-Max conditions and remember how these conditions were derived.

**SOLUTION:** For quantization levels \( q_k \) and bin dividers \( x_k \), we have Loyd-Max:

\[
x_k = \frac{q_{k+1} + q_k}{2}
\]

and

\[
q_k = E[X | X \in [x_{k-1}, x_k)]
\]

We need to prove that \( q_{k+1} \geq q_k \) or

\[
E[X | X \in [x_k, x_{k+1}]) - E[X | X \in [x_{k-1}, x_k]) \geq 0
\]

Well, by definition, the bin-dividers \( x_k \) are ordered from smallest to largest. That is \( x_{k+1} \geq x_k \). *Therefore, the intervals \([x_{k-1}, x_k)\) and \([x_k, x_{k+1})\) are consecutive and disjoint. Thus, for any PDF on \( X \) we must have*

\[
E[X | X \in [x_{k-1}, x_k)] \in [x_{k-1}, x_k)
\]

and

\[
E[X | X \in [x_{k-1}, x_k)] \in [x_{k-1}, x_k)
\]

which immediately implies

\[
E[X | X \in [x_{k-1}, x_{k+1}]) \geq E[X | X \in [x_{k-1}, x_k)]
\]

and thus \( q_{k+1} \geq q_k \).

3. (50 points) Cora and the Quantizer/Coder From Hell:

Cora the communications engineer has been hired by Mephisto Incorporated to design a communications hot line (tee hee) for the Prince of Darkness himself. The Prince wishes to use the hot line to remotely measure the temperature, \( x(t) \) in various parts of his domain. As
one might imagine, the sample sequence for the temperature is constantly increasing. In fact in one particular area the sampled temperature follows

\[ x_n = n/2 \]

Assume Cora needs to encode and transmit this sequence.

Cora has a choice of two systems. The block diagram for the first scheme is given in FIGURE 1. Basically, a direct difference is computed for the input signal \( x_n \) and input to a 1-bit quantizer. A coder then outputs a binary 1 or 0 depending upon whether the quantizer output is +1 or −1 respectively. At the receiver, the 1’s and 0’s are converted into ±1s and cumulatively summed to obtain \( \hat{x}_n \).

The block diagram for the second system is shown in FIGURE 2.

In this problem we will evaluate the effectiveness of both systems. For all parts assume that \( \hat{y}_0 = 1 \) and \( q_n = 0 \) for \( n < 0 \).

(a) (10 points) For system A in FIGURE 1, sketch the discrete sequence \( \hat{y}_n \) for \( n = 0, 1, \ldots, 10 \). What is the corresponding binary code sequence?

**SOLUTION:** The first coder only codes the difference directly, and the difference is always positive. Thus, the coder output is \( \hat{y}_n = 1 \) and the binary output is 111111... .

(b) (10 points) For system B in FIGURE 2, write down expressions for \( y_n \) and \( q_n \) by analyzing the block diagram and then sketch the discrete sequence \( \hat{y}_n \) for \( n = 0, 1, \ldots, 10 \). What is the corresponding binary code sequence?

**HINT:** It might help to put everything in a table.

**SOLUTION:** This coder does not blindly look at the difference. It takes into account the errors made by the coder by trying to predict \( x_{n−1} \) and transmit that difference; i.e., you have a negative feedback loop!

The equations we need to turn the crank are:

\[ y_n = x_n - q_{n−1} \]

where \( q_{n−1} \) is the input to the first adder and

\[ q_n = \hat{y}_n + q_{n−1} \]

Assuming initial rest (\( q_n = 0 \) for \( n < 0 \)) we have:
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(c) (10 points) For system A, carefully sketch the resulting $\hat{x}_n$. You may assume that $\hat{x}_0 = 0$.

**SOLUTION:** Sketch here is simple. Only 1’s are transmitted and that corresponds to always adding increments of +1. So $\hat{x}_n = n$.

(d) (10 points) Repeat the previous part for system B. Comment on any differences you find between the outputs generated by the two methods. Which, if either, does a better job? Why?

**SOLUTION:** In words, system B works out to, 3-up, 1-down, 3-up, 1-down for an average of 2-up every four steps; i.e., a slope of 1/2 just like we want.
Figure 1: System A for problem 3

Figure 2: System B for problem 3