ECE 330:541, Stochastic Signals and Systems
Homework 1 Supplemental Problem Solutions

Fall 2003

Supplemental Problem Solutions:

1. Another way to show that a sequence converges is to show that the sequence is a Cauchy sequence, and then use the fact that a sequence of real numbers converges if and only if it is a Cauchy sequences.

A sequence $\{x_n\}$ of real numbers is said to be a Cauchy sequence if for each $\epsilon > 0$, there exists an N (depending on ϵ) such that $|x_n - x_m| < \epsilon$ for all $n, m > N$.

Show that if a sequence $\{x_n\}$ satisfies $|x_{n+1} - x_n| \leq \alpha |x_n - x_{n-1}|$ for $n = 2, 3, \cdots$, and for some fixed $0 < \alpha < 1$, then $\{x_n\}$ is a convergent sequence.

Proof: Let $y = |x_2 - x_1|$. Then $|x_3 - x_2| \leq \alpha y$, and $|x_4 - x_3| \leq \alpha^2 y$. In general

$$|x_{n+1} - x_n| \leq y\alpha^{n-1}.$$

Now, choose n and m as in the definition of a Cauchy sequence. Without loss of generality, take $m > n$. Then, by the triangle inequality

$$|x_m - x_n| \leq \sum_{i=1}^{m-n} |x_{n+i} - x_{n+i-1}| \leq y \sum_{i=1}^{m-n} \alpha^{n+i-2} \leq \frac{y}{1-\alpha}\alpha^{n-1}.$$

This holds for all n and m and since $0 < \alpha < 1$, $|x_m - x_n| \to 0$ as n increases. Hence the sequence is Cauchy, and therefore it is a convergent sequence.

2. Suppose that $\{A_j\}$ is an infinite collection of disjoint events in a sample space Ω. Let P be a probability measure defined on Ω. Show that

$$\lim_{n\to\infty} P(A_j) = 0.$$

Proof: Observe that

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} A_j \leq 1.$$

Also observe that the partial sums

$$S_n = \sum_{j=1}^{n} A_j$$

form an increasing sequence. Since an increasing (monotonic) sequence that is bounded must converge, we have that $\sum_{j=1}^{\infty} A_j$ is a convergent series. Therefore, by properties of convergent series, the individual terms in the series must converge to 0, i.e. $P(A_j) \to 0$, which is the desired result.

3. Suppose A is a *null event*, that is, an event such that $P(A) = 0$. Show that

$$P(B \cup A) = P(B - A) = P(B)$$

for every event B in the sample space Ω. Here $B - A = B \cap A^c$.

Proof: The following relationship can be inferred:

$$P(B) \leq P(B \cup A) = P\left((B - A) \cup A\right) \leq P(B - A) + P(A) = P(B - A) \leq P(B).$$

Hence $P(B) \leq P(B \cup A) \leq P(B)$ which gives $P(B) = P(B \cup A)$.

4. The following is an example that pairwise independence does not mean independent. Suppose you have a container with 4 tickets, numbered 1234, 2341, 3412, and 4123. Suppose a ticket is drawn. Define the events

A: First digit of the ticket drawn is 1 or 4
B: Second digit of the ticket drawn is 2 or 4
C: Third digit of the ticket drawn is 3 or 4

Show that A, B, and C are pairwise independent, but collectively are not independent.

Proof:

Clearly $P(A) = P(B) = P(C) = \frac{1}{2}$, and $P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$, and finally $P(A \cap B \cap C) = \frac{1}{4}$. Pairwise independence follows from checking $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, and $P(B \cap C) = P(B)P(C)$. However $P(A \cap B \cap C) \neq P(A)P(B)P(C)$, and so the events are not collectively independent.