\[ \bar{\lambda}_i = \lambda_i \quad \text{for } i < k \]
\[ \bar{\lambda}_k = \frac{1 - \rho_1 - \cdots - \rho_{k-1}}{X_k} \]

For priorities \( i < k \) the arrival process is Poisson so the same calculation for the waiting time as before gives

\[ \sum_{i=1}^{k} \bar{\lambda}_i X_i \quad W_i = \frac{2(1 - \rho_1 - \cdots - \rho_{k-1})(1 - \rho_1 - \cdots - \rho_k)}{i < k} \]

For priority \( k \) and above we have infinite average waiting time in queue.

3.40

(a) The algebraic verification using Eq. (3.79) listed below

\[ W_k = \frac{R}{(1 - \rho_1 - \cdots - \rho_{k-1})(1 - \rho_1 - \cdots - \rho_k)} \]

is straightforward. In particular by induction we show that

\[ \rho_1 W_1 + \cdots + \rho_k W_k = \frac{R(\rho_1 + \cdots + \rho_k)}{1 - \rho_1 - \cdots - \rho_k} \]

The induction step is carried out by verifying the identity

\[ \rho_1 W_1 + \cdots + \rho_k W_k + \rho_{k+1} W_{k+1} = \frac{R(\rho_1 + \cdots + \rho_k)}{1 - \rho_1 - \cdots - \rho_k} + \frac{\rho_{k+1} R}{(1 - \rho_1 - \cdots - \rho_k)(1 - \rho_1 - \cdots - \rho_{k+1})} \]

The alternate argument suggested in the hint is straightforward.

(b) Cost

\[ C = \sum_{k=1}^{n} c_k N_k^k = \sum_{k=1}^{n} c_k \bar{\lambda}_k W_k = \sum_{k=1}^{n} \frac{c_k}{X_k} \rho_k W_k \]

We know that \( W_1 \leq W_2 \leq \cdots \leq W_n \). Now exchange the priority of two neighboring classes \( i \) and \( j = i+1 \) and compare \( C \) with the new cost

\[ C' = \sum_{k=1}^{n} \frac{c_k}{X_k} \rho_k W_k \]
In $C'$ all the terms except $k = i$ and $j$ will be the same as in $C$ because the interchange does not affect the waiting time for other priority class customers. Therefore

$$C' - C = \frac{c_i}{X_j} \rho_j W'_j + \frac{c_j}{X_i} \rho_i W'_i - \frac{c_i}{X_j} \rho_i W_j - \frac{c_j}{X_i} \rho_j W_i.$$ 

We know from part (a) that

$$\sum_{k=1}^{n} \rho_k W_k = \text{constant}.$$ 

Since $W_k$ is unchanged for all $k$ except $k = i$ and $j (= i+1)$ we have

$$\rho_i W_i + \rho_j W_j = \rho_i W'_i + \rho_j W'_j.$$ 

Denote

$$B = \rho_i W'_i - \rho_i W_i = \rho_j W'_j - \rho_j W_j.$$ 

Clearly we have $B \geq 0$ since the average waiting time of customer class $i$ will be increased if class $i$ is given lower priority. Now let us assume that

$$\frac{c_i}{X_i} \leq \frac{c_j}{X_j}.$$ 

Then

$$C' - C = \frac{c_i}{X_i} (\rho_i W'_i - \rho_i W_i) - \frac{c_j}{X_j} (\rho_j W'_j - \rho_j W_j) = B \left( \frac{c_i}{X_i} - \frac{c_j}{X_j} \right).$$ 

Therefore, only if $\frac{c_i}{X_i} < \frac{c_{i+1}}{X_{i+1}}$ can we reduce the cost by exchanging the priority order of $i$ and $i+1$. Thus, if $(1,2,3,...,n)$ is an optimal order we must have

$$\frac{c_1}{X_1} \geq \frac{c_2}{X_2} \geq \frac{c_3}{X_3} \geq ... \geq \frac{c_n}{X_n}.$$ 

3.41

Let $D(t)$ and $T_i(t)$ be as in the solution of Problem 3.31. The inequality in the hint is evident from Figure 3.30, and therefore it will suffice to show that
We have

\[ W = \frac{R}{1 - \rho} \]

where

\[ R = \lim_{t \to \infty} \left\{ \frac{1}{t} \sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 + \frac{1}{t} \sum_{i=1}^{L(t)} \frac{V_i^2}{2} \right\} \]

where \( L(t) \) is the number of vacations (or busy periods) up to time \( t \). The average length of an idle period is

\[ I = \int_0^\infty p(v) \left[ \int_0^v \nu \lambda e^{-\lambda \tau} d\tau + \int_v^\infty \tau \lambda e^{-\lambda \tau} d\tau \right] dv \]

and it can be seen that the steady-state time average number of vacations per unit time

\[ \lim_{t \to \infty} \frac{L(t)}{t} = \frac{1 - \rho}{I} \]

We have

\[ \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{L(t)} \frac{V_i^2}{2} = \lim_{t \to \infty} \frac{L(t)}{t} \frac{1}{2} \sum_{i=1}^{L(t)} \frac{V_i^2}{2} = \lim_{t \to \infty} \frac{L(t)}{t} \frac{V^2}{2I} = \frac{V^2 (1 - \rho)}{2I} \]

Therefore

\[ R = \frac{\lambda X^2}{2} + \frac{V^2 (1 - \rho)}{2I} \]

and

\[ W = \frac{\lambda X^2}{2 (1 - \rho)} + \frac{V^2}{2I} \]

3.47

(a) Since arrival times and service times are independent, the probability that there was an arrival in a small interval \( \delta \) at time \( \tau - x \) and that this arrival is still being served at time \( \tau \) is \( \lambda \delta [1 - F_X(x)] \).
(b) We have

\[ \bar{X} = \int_{0}^{\infty} x dF_X(x) \]

and by calculating the shaded area of the figure below in two different ways we obtain

\[ \int_{0}^{\infty} x dF_X(x) = \int_{0}^{\infty} [1 - F_X(x)] dx \]

This proves the desired expression.

(c) Let \( p_n(x) \) be the steady state probability that the number of arrivals that occurred prior to time \( \tau - x \) and are still present at time \( \tau \) is exactly \( n \).

For \( n \geq 1 \) we have

\[ p_n(x - \delta) = (1 - \lambda[1 - F_X(x)]\delta) p_n(x) + \lambda[1 - F_X(x)]\delta p_{n-1}(x) \]

and for \( n = 0 \) we have

\[ p_0(x - \delta) = (1 - \lambda[1 - F_X(x)]\delta) p_0(x) . \]

Thus \( p_n(x) \), \( n = 0, 1, 2, \ldots \) are the solution of the differential equations

\[ \frac{dp_n}{dx} = a(x)p_n(x) - a(x)p_{n-1}(x) \]

for \( n \geq 1 \).
\[ \frac{dp_0}{dx} = a(x)p_0(x) \]

where

\[ a(x) = \lambda [1 - F_X(x)]. \]

Using the known conditions

\[ p_n(\infty) = 0 \quad \text{for } n \geq 1 \]
\[ p_0(\infty) = 1 \]

it can be verified by induction starting with \( n = 0 \) that the solution is

\[ p_n(x) = \left[ \int_{a(y)dy}^{\infty} \right]_0^x \frac{[\int_{0}^{\infty} a(y)dy]^n}{n!}, \quad x \geq 0, \ n = 0, 1, 2, \ldots \]

Since

\[ \int_{0}^{\infty} a(y)dy = \lambda \int_{0}^{\infty} [1 - F_X(x)]dy = \lambda E(X) \]

we obtain

\[ p_n(0) = e^{-\lambda E(X)} \frac{[\lambda E(X)]^n}{n!}, \quad n = 0, 1, 2, \ldots \]

Thus the number of arrivals that are still in the system have a steady state Poisson distribution with mean \( \lambda E(X) \).

3.48

(a) Denote

\[ f(x) = E_r[(\max(0,r-x))^2] \]

and

\[ g(x) = (E_r[\max(0,r-x)])^2. \]

where \( E_r[-] \) denotes expected value with respect to \( r \) (\( x \) is considered constant). We will prove that \( f(x)/g(x) \) is monotonically nondecreasing for \( x \) nonnegative and thus attain its minimum value for \( x=0 \). We have