$$\tilde{\lambda}_i = \lambda_i \qquad \text{for } i < k$$

$$\tilde{\lambda}_k = \frac{(1 - \rho_1 - \cdots - \rho_{k-1})}{\bar{X}_k}$$

For priorities $i < k$ the arrival process is Poisson so the same calculation for the waiting time as before gives

$$W_i = \frac{\sum_{i=1}^{k} \tilde{\lambda}_i \overline{X_i^2}}{2(1 - \rho_1 - \cdots - \rho_{i-1})(1 - \rho_1 - \cdots - \rho_i)}, \qquad i < k$$

For priority k and above we have infinite average waiting time in queue.

3.40

(a) The algebraic verification using Eq. (3.79) listed below

$$W_k = R/(1 - \rho_1 - \ldots - \rho_{k-1})(1 - \rho_1 - \ldots - \rho_k)$$

is straightforward. In particular by induction we show that

$$\rho_1 W_1 + \cdots + \rho_k W_k = \frac{R(\rho_1 + \quad + \rho_k)}{1 - \rho_1 - \cdots - \rho_k}$$

The induction step is carried out by verifying the identity

$$\rho_1 W_1 + \cdots + \rho_k W_k + \rho_{k+1} W_{k+1} = \frac{R(\rho_1 + \cdots + \rho_k)}{1 - \rho_1 - \cdots - \rho_k} + \frac{\rho_{k+1} R}{(1 - \rho_1 - \cdots \rho_k)(1 - \rho_1 - \cdots - \rho_{k+1})}$$

The alternate argument suggested in the hint is straightforward.

(b) Cost

$$C = \sum_{k=1}^{n} c_k N_Q^k = \sum_{k=1}^{n} c_k \lambda_k W_k = \sum_{k=1}^{n} \left(\frac{c_k}{\overline{X}_k}\right) \rho_k W_k$$

We know that $W_1 \le W_2 \le \ldots\ldots \le W_n$. Now exchange the priority of two neighboring classes i and j=i+1 and compare C with the new cost

$$C' = \sum_{k=1}^{n} \left(\frac{c_k}{\overline{X}_k}\right) \rho_k W'_k$$

In C' all the terms except k = i and j will be the same as in C because the interchange does not affect the waiting time for other priority class customers. Therefore

$$C'-C = \frac{c_j}{\overline{\overline{X}}_j}\rho_j W'_j + \frac{c_i}{\overline{\overline{X}}_i}\rho_i W'_i - \frac{c_i}{\overline{\overline{X}}_i}\rho_i W_i - \frac{c_j}{\overline{\overline{X}}_j}\rho_j W_j.$$

We know from part (a) that

$$\sum_{k=1}^{n} \rho_k W_k = \text{constant.}$$

Since W_k is unchanged for all k except k = i and j (= i+1) we have

$$\rho_i W_i + \rho_j W_j = \rho_i W'_i + \rho_j W'_j.$$

Denote

$$B = \rho_i W'_i - \rho_i W_i = \rho_j W_j - \rho_j W'_j$$

Clearly we have $B \geq 0$ since the average waiting time of customer class i will be increased if class i is given lower priority. Now let us assume that

$$\frac{c_i}{\overline{\overline{X}}_i} \leq \frac{c_j}{\overline{\overline{X}}_j}$$

Then

$$C'-C = \frac{c_i}{\overline{\overline{X}}_i}(\rho_i W'_i - \rho_i W_i) - \frac{c_j}{\overline{\overline{X}}_j}(\rho_j W_j - \rho_j W'_j) = B\left(\frac{c_i}{\overline{\overline{X}}_i} - \frac{c_j}{\overline{\overline{X}}_j}\right)$$

Therefore, only if $\frac{c_i}{\overline{\overline{X}}_i} < \frac{c_{i+1}}{\overline{\overline{X}}_{i+1}}$ can we reduce the cost by exchanging the priority order of i and i+1. Thus, if (1,2,3,...,n) is an optimal order we must have

$$\frac{c_1}{\overline{\overline{X}}_1} \geq \frac{c_2}{\overline{\overline{X}}_2} \geq \frac{c_3}{\overline{\overline{X}}_3} \geq \ldots \geq \frac{c_n}{\overline{\overline{X}}_n}$$

3.41

Let D(t) and $T_i(t)$ be as in the solution of Problem 3.31. The inequality in the hint is evident from Figure 3.30, and therefore it will suffice to show that

3.46

We have

$$W = R/(1 - \rho)$$

where

$$R = \lim_{t\to\infty}\left\{\frac{1}{t}\sum_{i=1}^{M(t)}\frac{1}{2}X_i^2 + \frac{1}{t}\sum_{i=1}^{L(t)}\frac{V_i^2}{2}\right\}$$

where L(t) is the number of vacations (or busy periods) up to time t. The average length of an idle period is

$$I = \int_0^\infty p(v)\left[\int_0^V v\lambda e^{-\lambda\tau}d\tau + \int_V^\infty \tau\lambda e^{-\lambda\tau}d\tau\right]dv$$

and it can be seen that the steady-state time average number of vacations per unit time

$$\lim_{t\to\infty}\frac{L(t)}{t} = \frac{1 - \rho}{I}$$

We have

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{L(t)}\frac{V_i^2}{2} = \lim_{t\to\infty}\frac{L(t)}{t}\frac{\sum_{i=1}^{L(t)}\frac{V_i^2}{2}}{L(t)} = \lim_{t\to\infty}\frac{L(t)}{t}\frac{\overline{V^2}}{2I} = \frac{\overline{V^2}(1 - \rho)}{2I}$$

Therefore

$$R = \frac{\lambda\overline{X^2}}{2} + \frac{\overline{V^2}(1 - \rho)}{2I}$$

and

$$W = \frac{\lambda\overline{X^2}}{2(1 - \rho)} + \frac{\overline{V^2}}{2I}$$

3.47

(a) Since arrival times and service times are independent, the probability that there was an arrival in a small interval δ at time $\tau - x$ and that this arrival is still being served at time τ is $\lambda\delta[1 - F_X(x)]$.

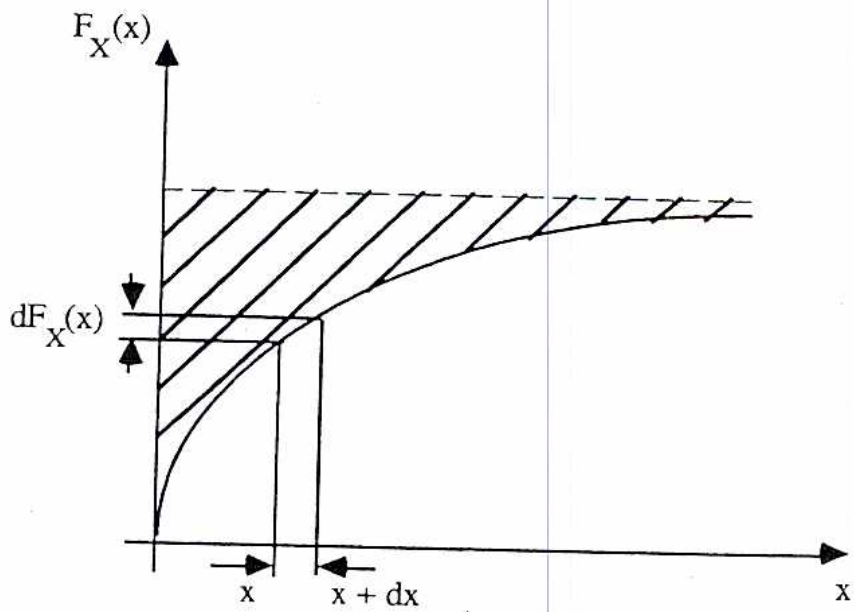
(b) We have

$$\overline{X} = \int_0^\infty x dF_X(x)$$

and by calculating the shaded area of the figure below in two different ways we obtain

$$\int_0^\infty x dF_X(x) = \int_0^\infty [1 - F_X(x)]dx$$

This proves the desired expression.



(c) Let $p_n(x)$ be the steady state probability that the number of arrivals that occurred prior to time $\tau - x$ and are still present at time τ is exactly n.

For $n \geq 1$ we have

$$p_n(x - \delta) = \{1 - \lambda[1 - F_X(x)]\delta\}p_n(x) + \lambda[1 - F_X(x)]\delta p_{n-1}(x)$$

and for n =0 we have

$$p_0(x - \delta) = \{1 - \lambda[1 - F_X(x)]\delta\}p_0(x).$$

Thus $p_n(x)$, n = 0, 1, 2, . . . are the solution of the differential equations

$$dp_n/dx = a(x)p_n(x) - a(x)p_{n-1}(x) \qquad \text{for } n \geq 1$$

$$dp_0/dx = a(x)p_0(x) \qquad \text{for } n = 0$$

where

$$a(x) = \lambda[1 - F_X(x)].$$

Using the known conditions

$$p_n(\infty) = 0 \qquad \text{for } n \geq 1$$
$$p_0(\infty) = 1$$

it can be verified by induction starting with n = 0 that the solution is

$$p_n(x) = [e^{-\int_x^\infty a(y)dy}] \frac{[\int_x^\infty a(y)dy]^n}{n!}, \qquad x \geq 0, \; n = 0, 1, 2, \ldots$$

Since

$$\int_0^\infty a(y)dy = \lambda \int_0^\infty [1 - F_X(x)]dy = \lambda E\{X\}$$

we obtain

$$p_n(0) = e^{-\lambda E\{X\}} \frac{[\lambda E\{X\}]^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

Thus the number of arrivals that are still in the system have a steady state Poisson distribution with mean $\lambda E\{X\}$.

3.48

(a) Denote

$$f(x) = E_r[(\max\{0, r-x\})^2]$$

and

$$g(x) = (E_r[\max\{0, r-x\}])^2,$$

where $E_r[\cdot]$ denotes expected value with respect to r (x is considered constant). We will prove that f(x)/g(x) is montonically nondecreasing for x nonnegative and thus attain its minimum value for x=0. We have