ECE Communication Networks 1
Homework 3 Solutions

Problems

1. **The $M/E_k/1$ queue:** Consider a queue with a Poisson arrival process, but where the service times are k-stage Erlangian. Here, the pdf for the service times are given by

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x}.$$

The Laplace transform of the density is given by

$$\hat{f}_X(s) = \left(1 + \frac{s}{k\mu}\right)^{-k}.$$

(You don't have to show the Laplace transform). Find the probability distribution $\{p_n\}$ for the amount of customers in the system.

Solution:

Recall from class that $A(z) = \hat{f}(\lambda - \lambda z)$, which gives

$$A(z) = \left[1 + \frac{\rho(1-z)}{k}\right]^{-k}.$$

Substituting into the expression for $P(z) = \frac{(1-\rho)(z-1)A(z)}{z-A(z)}$ gives

$$P(z) = \frac{(1-\rho)(z-1)}{z\left[1 + \frac{\rho(1-z)}{k}\right]^k - 1}.$$

Now, several people in class noticed that its hard to do something with this. Well, for small k you can use partial-fractions. There is a trick algebra substitution that helps simplify things. Let

$$U = 1 + \frac{\rho}{k}$$

and

$$V = \frac{\rho}{k+\rho} = 1 - U^{-1}.$$

Then

$$P(z) = \frac{(1-\rho)(1-z)}{1 - z[U(1-Vz)]^k}$$

which is

$$(1-\rho)(1-z)\sum_{j=0}^{\infty} z^j U^{kj}(1-Vz)^{kj}.$$

So, we have a series, and all we need to do is equate like to get p_n (its still a bit of a pain).

The trick is to use binomial expansion of $(1-Vz)^{kj}$ and realize that, for the calculation of p_n, we need to keep the $n-j$ and $n-j-1$ terms (corresponding to the $1-z$) for the j-th summand.

Here's what we get:

$$p_n = (1-\rho)\sum_{j=0}^{n}(-1)^{n-j}V^{n-j-1}U^{kj}\left[\binom{kj}{n-j} + \binom{kj}{n-j-1}\right].$$

There's not much more you can do with this.

2. **Alternate Derivation of Pollaczek-Khinchine:** Use the probability generating function expression for $P(z)$.

Recall that the expected value of a random variable can be obtained from evaluating the derivative of its probability generating function. Thus $E[N]$ can be obtained from the $P'(1)$. We use the fact that $\hat{f}'_X(s) = -E[X]$, and $\hat{f}_X(0) = 1$.

Take the derivative and substitute to get

$$E[N] = P'(1) = \rho + \frac{\lambda^2 E[X^2]}{2(1-\rho)}.$$