Problems
5. Consider the transition probability matrix

\[
P = \begin{pmatrix}
0.8 & 0.2 & 0.0 \\
0.8 & 0.0 & 0.2 \\
0.0 & 0.8 & 0.2
\end{pmatrix}
\]

for a discrete time Markov Chain. (a) Draw the Markov chain, and (b) compute the steady-state vector using at least 2 different approaches.

(a) The picture should have three states: 1, 2, and 3. There is a 1 to 1 flow of 0.8, and a 1 to 2 flow of 0.2. There is a 2 to 1 flow of 0.8, and a 2 to 3 flow of 0.2. There is a 3 to 3 flow of 0.2 and a 3 to 2 flow of 0.8.

(b) Call the matrix \( P \). One way to calculate the steady state vector is to look at \( P^k \). A quick calculation gives

\[
P^{16} = \begin{pmatrix}
0.762 & 0.190 & 0.048 \\
0.762 & 0.190 & 0.048 \\
0.762 & 0.190 & 0.048
\end{pmatrix}
\]

(You should be able to do this in less than 16 matrix multiplications). Hence the steady state \( \pi = [0.762, 0.190, 0.048] \). Another way is to solve \( \pi = \pi P \) as an eigenvalue-eigenvalue problem (in fact, the \( P^k \) method is just one way of solving an eigenvector problem).

A third method is to realize that this Markov chain describes a birth-death process, where the birth rate is 0.2 and the death rate is 0.8. The traffic intensity is \( \rho = 0.2/0.8 = 1/4 \), and

\[
\pi_1 = \frac{1 - 1/4}{1 - 1/4^3} = 16/21 = 0.762 \\
\pi_2 = \pi_1 \rho = 4/21 = 0.190 \\
\pi_3 = \pi_2 \rho = 1/21 = 0.048.
\]

6. (HARD) Consider a renewal (birth) process with interarrival times \( \tau_j \). Let \( W_0 = 0 \), and generally define the waiting time \( W_n = \sum_{k=1}^{n} \tau_k \) for \( n \geq 1 \). Associate the counting process \( N(t) \) with the renewal process, where \( N(t) \) is the number of events that have occurred between time 0 and time \( t \) (where \( N(0) = 0 \)). Assume the interarrival times are i.i.d. and follow a pdf \( f_{\tau}(t) \). Let \( \phi_{\tau}(z) \) be the probability generating function of \( f_{\tau}(t) \), then show that

\[
Pr[W_n \leq t] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\phi^n_{\tau}(z)}{z} e^{zt} dz.
\]

The waiting time \( W_n \) is the sum of \( n \) i.i.d random variables, and hence

\[
\phi_{W_n}(z) = (\phi_{\tau}(z))^n
\]

Now, we are interested in the cdf \( F_{W_n}(t) = Pr[W_n \leq t] = \int_0^t f_{W_n}(u) du \). Take the Laplace transform

\[
\int_0^\infty e^{-zt} F_{W_n}(t) dt = \frac{\phi_{W_n}(z)}{z} = \frac{\phi^n_{\tau}(z)}{z}
\]

where we have used properties of the Laplace transform to achieve the simplification. Now, all we need is to take the inverse Laplace transform in order to obtain:

\[
Pr[W_n \leq t] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\phi^n_{\tau}(z)}{z} e^{zt} dz.
\]