ECE Communication Networks 1
Homework Assignment 2

Fall 2006

Problems

5. Consider the transition probability matrix

$$P = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.8 & 0.0 & 0.2 \\ 0.0 & 0.8 & 0.2 \end{pmatrix}$$

for a discrete time Markov Chain. (a) Draw the Markov chain, and (b) compute the steady-state vector using at least 2 different approaches.

(a) The picture should have three states: 1, 2, and 3. There is a 1 to 1 flow of 0.8, and a 1 to 2 flow of 0.2. There is a 2 to 1 flow of 0.8, and a 2 to 3 flow of 0.2. There is a 3 to 3 flow of 0.2 and a 3 to 2 flow of 0.8.

(b) Call the matrix P. One way to calculate the steady state vector is to look at P^k. A quick calculation gives

$$P^{16} = \begin{pmatrix} 0.762 & 0.190 & 0.048 \\ 0.762 & 0.190 & 0.048 \\ 0.762 & 0.190 & 0.048 \end{pmatrix}$$

(You should be able to do this in less than 16 matrix multiplications). Hence the steady state $\pi = [0.762, 0.190, 0.048]$. Another way is to solve $\pi = \pi P$ as an eigenvalue-eigenvalue problem (in fact, the P^k method is just one way of solving an eigenvector problem).

A third method is to realize that this Markov chain describes a birth-death process, where the birth rate is 0.2 and the death rate is 0.8. The traffic intensity is $\rho = 0.2/0.8 = 1/4$, and

$$\pi_1 = \frac{1 - 1/4}{1 - 1/4^3} = 16/21 = 0.762$$

$$\pi_2 = \pi_1 \rho = 4/21 = 0.190$$

$$\pi_3 = \pi_2 \rho = 1/21 = 0.048.$$

6. (HARD) Consider a renewal (birth) process with interarrival times τ_j. Let $W_0 = 0$, and generally define the waiting time $W_n = \sum_{k=1}^{n} \tau_k$ for $n \geq 1$. Associate the counting process $N(t)$ with the renewal process, where $N(t)$ is the number of events that have occurred between time 0 and time t (where $N(0) = 0$). Assume the interarrival times are i.i.d. and follow a pdf $f_\tau(t)$. Let $\phi_\tau(z)$ be the probability generating function of $f_\tau(t)$, then show that

$$Pr[W_n \leq t] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\phi_\tau^n(z)}{z} e^{zt} dz.$$

The waiting time W_n is the sum of n i.i.d random variables, and hence

$$\phi_{W_n}(z) = (\phi_\tau(z))^n$$

Now, we are interested in the cdf $F_{W_n}(t) = Pr[W_n \leq t] = \int_0^t f_{W_n}(u) du$. Take the Laplace transform

$$\int_0^\infty e^{-zt} F_{W_n}(t) dt = \frac{\phi_{W_n}(z)}{z} = \frac{\phi_\tau^n(z)}{z}$$

where we have used properties of the Laplace transform to achieve the simplification. Now, all we need is to take the inverse Laplace transform in order to obtain:

$$Pr[W_n \leq t] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\phi_\tau^n(z)}{z} e^{zt} dz.$$