and the expected time to the next depature is 1/4. So the answer is 5/4 minutes.

(c) The answer will not change because the situation at the instant when the customer begins service will be the same under the conditions for (a) and the conditions for (c).

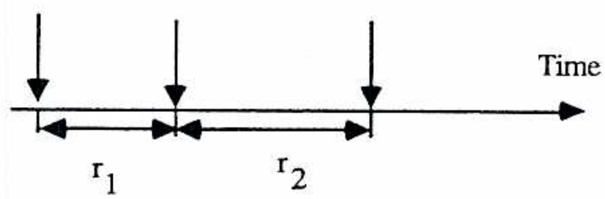
3.7

In the statistical multiplexing case the packets of at most one of the streams will wait upon arrival for a packet of the other stream to finish transmission. Let W be the waiting time , and note that $0 \le W \le T/2$. We have that one half of the packets have system time $T/2 + W$ and waiting time in queue W. Therefore

$$\text{Average System Time} = (1/2)T/2 + (1/2)(T/2+W) = (T+W)/2$$
$$\text{Average Waiting Time in Queue} = W/2$$
$$\text{Variance of Waiting Time} = (1/2)(W/2)^2+(1/2)(W/2)^2 = W^2/4.$$

So the average system time is between T/2 and 3T/4 and the variance of waiting time is between 0 and $T^2/16$.

3.8



Fix a packet. Let r_1 and r_2 be the interarrival times between a packet and its immediate predecessor, and successor respectively as shown in the figure above. Let X_1 and X_2 be the lengths of the predecessor packet, and of the packet itself respectively. We have:

$$P\{\text{No collision w/ predecessor or successor}\} = P\{r_1 > X_1, r_2 > X_2\}$$
$$= P\{r_1 > X_1\}P\{r_2 > X_2\}.$$

$$P\{\text{No collision with any other packet}\} = P_1\, P\{r_2 > X_2\}$$

where

$$P_1 = P\{\text{No collision with all preceding packets}\}.$$

(a) For fixed packet lengths (= 20 msec)

$$P\{r_1 > X_1\} = P\{r_2 > X_2\} = e^{-\lambda*20} = e^{-0.01*20} = e^{-0.2}$$
$$P_1 = P\{r_1 \le X_1\}.$$

Therefore the two probabilities of collision are both equal to $e^{-0.4} = 0.67$.

(3.11)

(a) Let us call the two transmission lines 1 and 2, and let $N_1(t)$ and $N_2(t)$ denote the respective numbers of packet arrivals in the interval $[0,t]$. Let also $N(t) = N_1(t) + N_2(t)$. We calculate the joint probability $P\{N_1(t) = n, N_2(t) = m\}$. To do this we first condition on $N(t)$ to obtain

$$P\{N_1(t) = n, N_2(t) = m\} = \Sigma_{k=0}^{\infty} P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\}P\{N(t) = k\}.$$

Since

$$P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\} = 0 \qquad \text{when} \quad k \neq n+m$$

we obtain

$$P\{N_1(t) = n, N_2(t) = m\} = P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\}P\{N(t) = n + m\}$$
$$= P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\}e^{-\lambda t}[(\lambda t)^{n+m}/(n + m)!]$$

However, given that n+m arrivals occurred, since each arrival has probability p of being a line 1 arrival and probability 1-p of being a line 2 arrival, it follows that the probability that n of them will be line 1 and m of them will be line 2 arrivals is the binomial probability

$$\binom{n+m}{n} p^n (1-p)^m$$

Thus

$$P\{N_1(t) = n, N_2(t) = m\} = \binom{n+m}{n} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!}$$

$$= e^{-\lambda tp} \frac{(\lambda tp)^n}{n!} \quad e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^m}{m!} \tag{1}$$

Hence

$$P\{N_1(t) = n\} = \sum_{m=0}^{\infty} P\{N_1(t) = n, N_2(t) = m\}$$

$$= e^{-\lambda tp} \frac{(\lambda tp)^n}{(n)!} \sum_{m=0}^{\infty} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^m}{m!}$$

$$= e^{-\lambda tp} \frac{(\lambda tp)^n}{(n)!}$$

That is, $\{N_1(t), t \geq 0\}$ is a Poisson process having rate λp. Similarly we argue that $\{N_2(t), t \geq 0\}$ is a Poisson process having rate $\lambda(1 - p)$. Finally from Eq. (1) it follows that the two processes are independent since the joint distribution factors into the marginal distributions.

(b) Let A, A_1, and A_2 be as in the hint. Let I be an interarrival interval of A_2 and consider the number of arrivals of A_1 that lie in I. The probability that this number is n is the probability of n successive arrivals of A_1 followed by an arrival of A_2, which is $\rho^n(1 - \rho)$. This is also the probability that a customer finds upon arrival n other customers waiting in an M/M/1 queue. The service time of each of these customers is exponentially distributed with parameter μ, just like the interarrival times of process A. Therefore the waiting time of the customer in the M/M/1 system has the same distribution as the interarrival time of process A_2. Since by part (a), the process A_2 is Poisson with rate $\mu - \lambda$, it follows that the waiting time of the customer in the M/M/1 system is exponentially distributed with parameter $\mu - \lambda$.

3.12

For any scalar s we have using also the independence of τ_1 and τ_2

$$P(\min\{\tau_1,\tau_2\} \geq s) = P(\tau_1 \geq s, \tau_2 \geq s) = P(\tau_1 \geq s)\, P(\tau_2 \geq s)$$

$$= e^{-\lambda_1 s}\, e^{-\lambda_2 s} = e^{-(\lambda_1 + \lambda_2)s}$$

Therefore the distribution of $\min\{\tau_1,\tau_2\}$ is exponential with mean $1/(\lambda_1 + \lambda_2)$.

By viewing τ_1 and τ_2 as the arrival times of the first arrivals from two independent Poisson processes fwith rates λ_1 and λ_2, we see that the equation $P(\tau_1 < \tau_2) = \lambda_1/(\lambda_1 + \lambda_2)$ follows from Problem 3.10(c).

Consider the M/M/1 queue and the amount of time spent in a state k>0 between transition into the state and transition out of the state. This time is $\min\{\tau_1,\tau_2\}$, where τ_1 is the time between entry to the state k and the next customer arrival and τ_2 is the time between entry to the state k and the next service completion. Because of the memoryless property of the exponential distribution, τ_1 and τ_2 are exponentially distributed with means $1/\lambda$ and $1/\mu$, respectively. It follows using the fact shown above that the time between entry and exit from stae k is exponentially distributed with mean $1/(\lambda+\mu)$. The probability that the transition will be from k to k+1 is $\lambda/(\lambda+\mu)$ and that the transition will be from k to k-1 is $\mu/(\lambda+\mu)$. For state 0 the amount of time spent is exponentially distributed with mean $1/\lambda$ and the probability of a transition to state 1 is 1. Because of this it can be seen that M/M/1 queue can be described as a continuous Markov chain with the given properties.

3.13

(a) Consider a Markov chain with state

n = Number of people waiting + number of empty taxi positions

Then the state goes from n to n+1 each time a person arrives and goes from n to n-1 (if $n \geq 1$) when a taxi arrives. Thus the system behaves like an M/M/1 queue with arrival rate 1 per min and departure rate 2 per min. Therefore the occupancy distribution is equiprobable states 0, 1', . . ., (k-1)' we obtain $p_0 = (1 - \rho)/k$. The average number in the system is

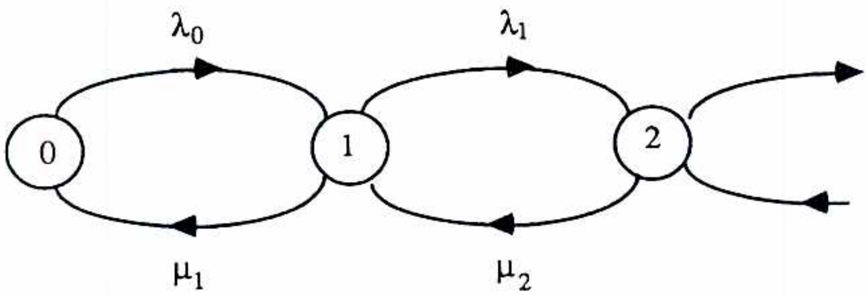
$$N = p_{1'} + 2p_{2'} + \ldots + (k-1)p_{(k-1)'} + \sum_{i=0}^{\infty} ip_i = p_0 \frac{k(k-1)}{2} + \sum_{i=0}^{\infty} ip_i$$

where the probabilities p_i are given in the equations above. After some calculation this yields

$$N = (k-1)/2 + \rho/(1-\rho).$$

The average time in the system is (by Little's Theorem) $T = N/\lambda$.

3.16



The figure shows the Markov chain corresponding to the given system. The local balance equation for it can be written down as :

$$\rho_0 p_0 = p_1$$

$$\rho_1 p_1 = p_2$$

$$\ldots \qquad \ldots$$

$$\Rightarrow p_{n+1} = \rho_n p_n = \rho_{n-1}\rho_n p_{n-1} = \ldots = (\rho_0 \rho_1 \cdots \rho_n) p_0$$

but,

$$\sum_{i=0}^{\infty} p_i = p_0(1 + \rho_0 + \rho_0\rho_1 + \ldots) = 1$$

$$\Rightarrow p_0 = \left[1 + \sum_{k=0}^{\infty} \left(\rho_0 \cdots \rho_k\right)\right]^{-1}$$

We have here an M/M/m/m system where m is the number of circuits provided by the company. Therefore we must find the smallest m for which $p_m < 0.01$ where p_m is given by the Erlang B formula

$$p_m = \frac{(\lambda/\mu)^m/m!}{\sum_{n=0}^{m}(\lambda/\mu)^n/n!}.$$

We have $\lambda = 30$ and $\mu = 1/3$, so $\lambda/\mu = 30\cdot 3 = 90$. By substitution in the equation above we can calculate the required value of m.

3.20

We view this as an M/M/m problem. We have

$\lambda=0.5$, $E(X) = 1/\mu = 3$, $m=?$ so that $W<0.5$

We know that the utilization factor has to be less than 1 or m has to be greater than or equal to 2. By the M/M/n results we have

$$W = \frac{\frac{\lambda}{m\mu} P_Q}{\lambda\,(1-\frac{\lambda}{m\mu})} = \frac{P_Q}{m\mu-\lambda}$$

$$\text{where } P_Q = \frac{p_0 \,(\frac{\lambda}{\mu})^m}{m!\,(1-\frac{\lambda}{m\mu})}$$

$$\text{and} \quad p_0 = \left[\sum_{n=0}^{m-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^m}{m!\,(1-\lambda/\mu)}\right]^{-1}$$

As can be seen from the expressions above m should be at most 5 because at m=5 , W is less than 0.5 because P_Q is less than 1.

The following C program calculates the optimum m.

```
double P0(lambda,mu,m){
        mrho = lambda/mu;
        rho = mrho/m;
        for(n=0; n<m; n++)
                temp1 = pow(mrho,n)/ fact(n);
```

```
        temp2 = pow(mrho,m)/(fact(m)*(1-rho));
        return(1/( temp1 + temp2 )); /* this returns p0 */
}

int fact(n){
        if (n==0) return (1);
        else
                return(n* fact (n-1));
}

double W(lambda,mu,m){
        PQ = P0(lambda,mu,m) * pow(mrho,m) /
                                        (fact(m) * (1-rho));
return(PQ/(m *mu - lambda));
}       /* this returns W for a given m */

main() {
        lambda = 0.5; mu = 0.333; previous_W = 100.0;
        for(m=2; m<=5; m++)
                if ((temp = W(lambda,mu,m)) < Previous_W)
                        previous_W = temp;
                else
                        { print(m-1);
                        break;
                }
}
```

3.21

We have $p_n = \rho^n p_0$ where $\rho = \lambda/\mu$. Using the relation

$$\sum_{n=0}^{m} p_n = 1$$

we obtain

$$p_0 = \frac{1}{\sum_{n=0}^{m} \rho^n} = \frac{1 - \rho}{1 - \rho^{m+1}}$$

Thus

$$p_n = \frac{\rho^n(1 - \rho)}{1 - \rho^{m+1}}, \qquad 0 \le n \le m$$

3.22

(a) When all the courts are busy, the expected time between two departures is 40/5 = 8 minutes. If a pair sees k pairs waiting in the queue, there must be exactly k+1 departures from the system before they get a court. Since all the courts would be busy during this whole time, the average waing time required before k+1 departures is 8(k+1) minutes.