ECE Communication Networks 1
Final Exam, Due: Dec 19th, 3pm

Fall 2006

Instructions: The exam is a takehome exam. You are to work on the exam alone. Please clearly indicate which problem you are solving and arrange the problems in order.

1. (**Honor Code:**) On a separate page at the front of the exam you must write out:

   `I attest that I have not given, or received any aid or discussion in relationship to this examination.`

   Additionally, you must sign this proclamation.

2. (**How Discrete!**) In this problem, we consider a discrete time analog of the M/M/1 queue. Consider the time axis divided into equal slots. Let the arrival process be a Bernoulli sequence with probability p (i.e. a discrete time "0-1" sequence X_k where $Pr[X_k = 1] = p$). Suppose the service time is geometrically distributed with parameter q (i.e. the service time S has the probability distribution $Pr[S = l] = q(1-q)^{l-1}$).

   In order to proceed, let us assume that: (i) a newly arriving customer is accepted into the system slightly before a new time slot begins; (ii) if the service is completed in a given time slot, the customer leaves the system before the next time slot starts; (iii) a customer waiting in the queue enters the service as soon as a previous customer leaves the server so that the new slot can be used.

   Define $N(k)$ to be the number of customers in the system at time slot k. Define $A(k) = 1$ if there is an arrival in time $[k, k+1)$ and $A(k) = 0$ otherwise. Similarly, define $D(k) = 1$ if there is a departure on time $[k, k+1)$ and $D(k) = 0$ otherwise.

   (a) (**5 points**) Show that the sequence $\{N(k)\}$ is a Markov sequence.

   (b) (**10 points**) Calculate the transition probability matrix

   $$P_{i,j} = Pr[N(k+1) = j | N(k) = i]$$

   (c) (**10 points**) Define the probability $\pi_i(k) = Pr[N(k) = i]$. If we define the matrix $\mathbf{P} = [P_{i,j}]$ and the vector $\pi(k) = [\pi_0(k), \pi_1(k), \cdots, \pi_i(k), \cdots]$.
   Then, $\pi(k)$ satisfies

   $$\pi(k+1) = \pi(k)\mathbf{P}.$$

   Let $\pi_i = \lim_{k\to\infty} \pi_i(k)$. Using this relationship, give the balance equations (note: there will be separate results for $i = 0$ and $i = 1, 2, \cdots$).

   (d) (**10 points**) Using the balance equations, solve for the steady state probability distribution π_i in terms of p and q.

3. (**This Ain't No English:**) Markov models are powerful tools, and in this problem you are to explore the use of the Markov process as a means to generate traffic. On the course website is a file for `Sample.txt`. This problem involves programming, and in order to get full credit, you must describe your methodology, implementation, and the *empirical* results you have obtained for your Markov model.

   (a) (**10 points**) One approach to modeling a language is to create a Markov model that describes the transition probabilities for going from one symbol to the next. For example, there are 26 letters in the English language, and we can build a 26×26 probability matrix describing the likelihood of going from one character to the next. Using the text provided, you are to build a single-letter and a digram (a digram is a two-letter combination) Markov model for the English language. To do this, you may use any programming language of your choice. Be sure to calculate both transition probabilities and steady-state probabilities for each letter/digram.

(b) (**10 points**) Similar to symbol-based traffic generation, we may also build word-based traffic generators. Using the text provided, build a word-based Markov model for the English language. Provide a list of words in your state-space, their associated probabilities, and the transition probabilities.

(c) (**10 points**) Using your single-letter, digram, and word-level Markov models, generate a sample of text for each model. The sample should be at least 100 characters long.

4. (**How General Can We Get?:**) Consider a single server system, for which we define the variables:

$$S_k = \text{Service time of customer } C_k$$

$$W_k = \text{Waiting time of } C_k$$

$$X_k = \text{Interarrival time between } C_{k-1} \text{ and } C_k.$$

Define $Y_k = S_k - X_{k+1}$

(a) (**5 points**) Show that $W_{k+1} = \max\{W_k + Y_k, 0\}$ by considering the cases when $W_k + S_k - X_{k+1} > 0$ and when $W_k + S_k - X_{k+1} < 0$.

(b) (**5 points**) Define

$$F_{W_{k+1}}(t) = \begin{cases} Pr[W_k + Y_k \leq t] & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases}$$

You may assume W_k and Y_k are independent. Let $F_W(t) = \lim_{k\to\infty} F_{W_k}(t)$. Show that

$$F_W(t) = \begin{cases} \int_0^\infty F_Y(t-u) f_W(u) du & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases}$$

and equivalently that

$$F_W(t) = \begin{cases} \int_{-\infty}^t F_W(t-y) f_Y(y) dy & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases}$$

(c) The above result for $F_W(t)$ is an integral equation of Wiener-Hopf type, and can be used to obtain the waiting time for a general single-server queueing system. One approach to solving such integral equations is by the method of spectral factorization. In spectral factorization, we define

$$F_{W-}(t) = \begin{cases} \int_{-\infty}^t F_W(t-y) f_Y(y) dy & \text{for } t < 0 \\ 0 & t \geq 0 \end{cases}$$

In what follows, you may assume that all distributions involved decay faster than the exponential (for integral convergence).

i. (**5 points**) Apply the double-sided Laplace Transform to obtain

$$\hat{f}_{W-}(s) = [\hat{f}_Y(s) - 1]\hat{f}_W(s),$$

where $\hat{f}_W(s)$ is the Laplace Transform of $F_W(t)$, and $\hat{f}_Y(s)$ is the Laplace Transform of $f_Y(t)$ (**note:** be careful, f_Y is the pdf, F_W is a cdf). Here, we also define

$$\hat{f}_{W-}(s) = \int_{-\infty}^0 e^{-st} F_{W-}(t) dt.$$

ii. (**5 points**) Applying the double-sided Laplace Transform, show $\hat{f}_Y(s) = \hat{f}_X(-s)\hat{f}_S(s)$, where $\hat{f}_X(s)$ and $\hat{f}_S(s)$ are the usual Laplace transforms of $F_X(t)$ and $F_S(t)$.

iii. (**5 points**) Suppose that you can factor

$$\hat{f}_Y(s) - 1 = \hat{f}_X(-s)\hat{f}_S(s) - 1 = \frac{A(s)}{B(s)}$$

where $A(s)$ is analytic and contains no zeros in the half plane $Re(s) > 0$, and $B(s)$ is analytic and contains no zeros in $Re(s) < \xi$. Further, suppose that $A(s)$ and $B(s)$ have the properties: $\lim_{|s|\to\infty} A(s) = s$ for $Re(s) > 0$, and $\lim_{|s|\to\infty} B(s) = -s$ for $Re(s) < \xi$. Then, it can be shown (you do NOT have to do this... it involves some heavy complex analysis) that

$$\hat{f}_W(s)A(s) = \hat{f}_{W-}(s)B(s)$$

for all s, and thus that

$$\hat{f}_W(s) = \frac{K}{A(s)}.$$

By looking at $K = \lim_{s\to\infty} \hat{f}_W(s)A(s)$, show $K = F_W(0)$.

iv. (**5 points**) Finally, show also that $F_W(0)$ can be found by

$$F_W(0) = \lim_{s\to\infty} \frac{A(s)}{s}.$$

(d) (**5 points**) Part (c) has just outlined a method for finding the waiting time distribution. Putting it all together, you calculate $\hat{f}_Y(s)$, factor $\hat{f}_Y(s) - 1$, solve for $F_W(0)$, to obtain $\hat{f}_W(s)$, and take the inverse Laplace Transform. Using the method just outlined, consider the $M/M/1$ queue (with arrival parameter λ and departure parameter μ), and show that the waiting time has distribution $F_W(t) = 1 - \rho e^{-(\mu-\lambda)t}$.