<u>ECE 545 Communication Theory Midterm</u>
<u>Problem # 3</u>

Let the transmitted signal be:

$$s(t) = \sum_{k=-\infty}^{k=\infty} a_k g(t - kT)$$

where $\{a_k\}$ is a WSS sequence of random real-valued symbols with a correlation function $R_a(k)$, *T* is the symboling interval, and *g(t)* is the symboling pulse with a Fourier transform *G(f)*, and the correlation function

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t) g(t + \tau) dt$$

Note that *s(t)* is not stationary but cyclo-stationary with a correlation function $R_s(t + \tau, t) = E\{s(t + \tau)s(t)\}$

(a) Compute the average correlation function of s(t) in terms of previously defined functions as follows:

$$\overline{R}_s(\tau) = 1/T \int_{-T/2}^{T/2} R_s(t + \tau, t) dt$$

(b) Express the power spectral density of *s(t)*, $S_s(f)$ in terms of previously introduced functions.

(c) Plot $S_s(f)$ for a white sequence a_k and g(t) = p(t) = 1 for $0 \le t \le T$ and *0* otherwise.

(d) Assume a white sequence a_k and a band limited power spectral density: $S_s(f) = 1$ for $-1/(2T) \le f \le 1/(2T)$ and *0* otherwise, compute:

$$\int_{-\infty}^{\infty} s(t) g(t) dt = ?$$

(e) Assume *r(t) = s(t) + n(t)* where n(t) is AWGN with spectral density $N_0/2$ and the conditions under (d). Compute the conditions (given transmitted symbols a_k) covariance matrix of the vector $[d_0 d_1]^T$ where

$$d_i = \int_{-\infty}^{\infty} r(t) g(t - iT) dt \quad \text{for} \quad i \in \{0,1\}$$

Solution

(a) Compute the average correlation function of s(t) in terms of previously defined functions as follows:

$$\overline{R}_s(\tau) = 1/T \int_{-T/2}^{T/2} R_s(t+\tau,t)dt$$

Solution:

$$R_s(t+\tau,t) = E\{s(t+\tau)s(t)\}$$

$$= \sum_{k=-\infty}^{k=\infty} \sum_{l=-\infty}^{k=\infty} E(a_k a_l) g(t-kT)g(t+\tau-lT)$$

$$= \sum_{m=-\infty}^{k=\infty} \sum_{l=-\infty}^{k=\infty} R_a(m) g(t-kT)g(t+\tau-lT)$$

$$= \sum_{m=-\infty}^{k=\infty} \sum_{l=-\infty}^{k=\infty} R_a(m) g(t-(l+m)T)g(t+\tau-lT)$$

$$\because k-l=m$$

$$\overline{R}_s(\tau) = 1/T \sum_{m=-\infty}^{m=\infty} R_a(m) \sum_{l=-\infty}^{l=\infty} \int_{-T/2}^{T/2} g(t-(l+m)T)g(t+\tau-lT)dt$$

$$\overline{R}_s(\tau) = 1/T \sum_{m=-\infty}^{m=\infty} R_a(m) \sum_{l=-\infty}^{l=\infty} \int_{-T/2+lT}^{T/2+lT} g(u-mT)g(u+\tau)du$$

$$\overline{R}_s(\tau) = 1/T \sum_{m=-\infty}^{m=\infty} R_a(m) \int_{-\infty}^{\infty} g(u-mT)g(u+\tau)du$$

$$= 1/T \sum_{m=-\infty}^{k=\infty} R_a(m) R_g(\tau+mT) \qquad (1)$$

(b) Express the power spectral density of *s(t)*, $S_s(f)$ in terms of previously introduced functions.
Solution:

$$S_s(f) \leftrightarrow \int_{-\infty}^{\infty} \overline{R_s}(\tau) e^{-j2\pi fm\tau} d\tau$$

$$R_g(\tau + mT) \leftrightarrow |G(f)|^2 \; e^{j2\pi f mT} \tag{2}$$

Substituting (2) in (1):

$$= \{(1/T)\sum_{m=-\infty}^{\infty} R_a(m)e^{j2\pi f mT}\} \, |G(f)|^2$$

$$= (1/T)S_a(f)\,|G(f)|^2$$

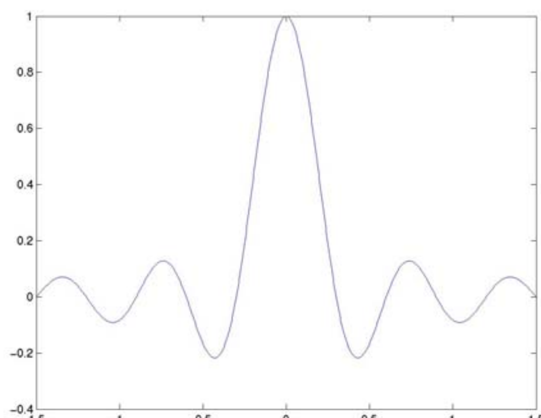
$$= S_s(f)$$

(c) Plot $S_s(f)$ for a white sequence a_k and g(t) = p(t) = 1 for $0 \le t \le T$ and *0* otherwise.

Solution: $S_s(f)$ = *(1/T)* $S_a(f)\,|G(f)|^2$

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt$$
$$= \int_{0}^{T} e^{-j2\pi ft}\,dt \because g(t) = 1$$
$$= e^{-j\pi ft}[\sin(\pi ft)/(\pi f)]$$
$$\therefore |G(f)|^2 = [\sin^2(\pi ft)/(\pi f)^2]$$

$S_a(f)$ is a constant having a non zero value only at *m=0,* say *L*. So then we can plot $S_s(f) = (L/T)\sin^2(\pi ft)/(\pi f)^2$ as follows:



(d) Assume a white sequence a_k and a band limited power spectral density: $S_s(f)$ =1 for $-1/(2T) \le f \le 1/(2T)$ and *0* otherwise, compute:

$$\int_{-\infty}^{\infty} s(t)g(t)dt = ?$$

Solution: $\int_{-\infty}^{\infty} s(t)g(t)dt = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{k=\infty} a_k g(t-kT)g(t)dt$

$$= \sum_{k=-\infty}^{k=\infty} a_k \int_{-\infty}^{\infty} g(t-kT)g(t)dt$$

$$= \sum_{k=-\infty}^{k=\infty} a_k R_g(kT)$$

$$R_g(\tau) = \int_{-\infty}^{\infty} (T/L)S_s(f)e^{j2\pi f\tau}df \because S_s(f) = 1, |f| < (1/2T)$$

$$= (T/L)\int_{-1/2T}^{1/2T} e^{j2\pi f\tau}df$$
$$= (T/L)[\sin(\pi\tau/T)/(\pi\tau)]$$

$$\therefore \int_{-\infty}^{\infty} s(t)g(t)dt = \sum_{k=-\infty}^{k=\infty} a_k R_g(kT)$$
$$= (T/L)\sum_{k=-\infty}^{k=\infty} a_k \sin(\pi k)/(\pi k)$$
$$= a_0(T/L)$$

(e) Assume *r(t) = s(t) + n(t)* where n(t) is AWGN with spectral density $N_0/2$ and the conditions under (d). Compute the conditions (given transmitted symbols a_k) covariance matrix of the vector $[d_0 d_1]^T$ where

$$d_i = \int_{-\infty}^{\infty} r(t)g(t-iT)dt \quad \text{for} \quad i \in \{0,1\}$$

Solution:

$$d_i = \int_{-\infty}^{\infty} (s(t)+n(t))g(t-iT)dt$$

$$d_i = (T/L)a_i + \int_{-\infty}^{\infty} n(t)g(t-iT)dt \qquad \text{from (3)}$$

$$E(d_i \mid \{a_i\}) = Ma_i \because M = T/L$$

$$E\{(d_i - Ma_i)^2 \mid \{a_i\}\} = E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(t)n(u)g(t-iT)g(u-iT)dtdu\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-u)(N_o/2)g(t-iT)g(u-iT)dtdu$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (N_o/2)g^2(u-iT)du$$

$$= (N_0/2)R_g(0)$$

$$= (N_0/2)(T/L)$$

$$E\{(d_i - Ma_iR_g(0))(d_j - Ma_jR_g(0)) \mid \{a_k\}\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{n(t)n(u)g(t-iT)g(u-iT)\}dtdu$$

$$= N_0/2\int_{-\infty}^{\infty} g(t-iT))g(u-jT)dtdu$$

$$= (N_0/2)R_g[(j-i)T]$$

$$= (N_0/2)(T/L) \quad \text{when } i=j \text{ and}$$

$$0 \quad \text{when } i \neq j$$