5.8

a) For Orthogonal signals, we have

$$C_N = \frac{1}{2}\log_2(1 + \frac{2E_N}{N_0})$$

Noting that

$$E_N = \frac{P_s}{D} = \frac{\text{energy/sec}}{\text{dim/sec}}$$

we get

$$C_N = \frac{1}{2}\log_2(1 + \frac{2P_s}{DN_0})$$

Thus, C_N decreases monotonically to 0 as D tends to ∞.

Capacity in bits per second is $C = DC_N$ and

$$\lim_{D\to\infty} C = \frac{1}{2\ln 2}\lim_{D\to\infty} D\ln(1 + \frac{2P_s}{N_0 D}) = \frac{P_s/N_0}{\ln 2}$$

also

$$\frac{\partial C}{\partial D} = D\frac{\partial C_N}{\partial D} + C_N$$

$$= \frac{1}{2\ln 2}(\ln(1+x) - \frac{x}{1+x}) \quad \text{where } x = \frac{2P_s}{DN_0}$$

Now

$$\ln(1+x) - \frac{x}{1+x} = -\left[\ln(\frac{1}{1+x}) + \frac{x}{1+x}\right]$$

recalling $\ln(a) < a - 1$ we see

$$-\left[\ln(\frac{1}{1+x}) + \frac{x}{1+x}\right] > -\left[(\frac{1}{1+x} - 1) + \frac{x}{1+x}\right] = 0$$

so

$$\frac{\partial C}{\partial D} > 0 \text{ for all } D > 0 \text{ and}$$

C increases monotonically to $P_s/N_0 \ln 2$ as D tends to ∞.

b) For the binary antipodal case

i) $R_0 = 1 - \log_2\left[1 + e^{-E_N/N_0}\right] = 1 - \log_2\left[1 + e^{-P_s/DN_0}\right]$

Thus R_0 decreases monotonically to 0 as D tends to ∞.

5.9 continued

ii) $R = DR_0$; let $x = P_s/DN_0$. Thus

$$\frac{\partial R}{\partial D} = R_0 + D\frac{\partial R_0}{\partial D} = \frac{1}{\ln 2}\left[\ln\frac{2}{1+e^{-x}} - \frac{x\,e^{-x}}{1+e^{-x}}\right]$$

$$= \frac{1}{\ln 2}\left[A(x) - B(x)\right]$$

However,

$$\frac{dA(x)}{dx} = \frac{e^{-x}}{1+e^{-x}}$$

$$\frac{dB(x)}{dx} = \frac{e^{-x}}{1+e^{-x}}\left[1 - \frac{x}{1+e^{-x}}\right]$$

so

$$\frac{dB(x)}{dx} < \frac{dA(x)}{dx} \quad \text{for all } x > 0$$

Since $A(0) = B(0) = 0$, we have $B(x) < A(x)$ for all $x > 0$ and

$$\frac{\partial R}{\partial D} = \frac{1}{\ln 2}\left[A(x) - B(x)\right] > 0, \; x > 0.$$

Thus DR_0 increases monotonically as D increases.

(ii) We desire a value of D such that

$$\frac{1}{2}C_\infty(1 - \alpha) \leqslant DR_0, \; 0 < \alpha << 1$$

Letting $x = P_s/DN_0 > 0$, and using $e^{-x} \leqslant 1 - x + x^2/2$ and $-\ln(1-W) > W + (W^2/2)$, $0 < W < 1$, we have

$$R_0 \ln 2 = -\ln\frac{1}{2}(1 + e^{-x})$$

$$> -\ln(1 - \frac{x}{2} + \frac{x^2}{4})$$

$$> \frac{x}{2} - \frac{x^2}{4} + \frac{1}{2}(\frac{x}{2} - \frac{x^2}{4})^2$$

$$> \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{8}$$

However,

5.8 continued

$$\left(\frac{\ln 2}{D}\right) \frac{1}{2} C_{\infty}(1 - \alpha) = \frac{1 - \alpha}{2} x$$

so it suffices to choose D so that

$$\frac{1 - \alpha}{2} x \leqslant \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{8}$$

or

$$\alpha \leqslant \frac{x}{4} - \frac{x^2}{4}$$

For α very small, x is very small, x^2 is negligible compared to x, and we choose D so that

$$\alpha \leqslant \frac{x}{4} = \frac{P_s}{4DN_0}$$

5.9

a) For k = 1 and 2, we have equality:

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 A_2]$$

Assume the bound is true for $P\left[\bigcup_{i=1}^{k-1} A_i\right]$. Then,

$$P\left[\bigcup_{i=1}^{k} A_i\right] = P\left[\left(\bigcup_{i=1}^{k-1} A_i\right) \cup A_k\right]$$

$$= P\left[\bigcup_{i=1}^{k-1} A_i\right] + P[A_k] - P\left[\bigcup_{j=1}^{k-1} A_k A_j\right]$$

$$\geq \sum_{i=1}^{k} P[A_i] - \sum_{i=2}^{k-1} \sum_{j=1}^{i-1} P[A_i A_j] - \sum_{j=1}^{k-1} P[A_k A_j]$$

$$= \sum_{i=1}^{k} P[A_i] - \sum_{i=2}^{k} \sum_{j=1}^{i-1} P[A_i A_j]$$

which completes the induction.

b) Let A_i be the event that $r_i \geqslant r_k$ given that m_k is transmitted. Then

$$P[\mathcal{E}] = P[\mathcal{E}|m_k] = P\left[\bigcup_{\substack{i=1 \\ i \neq k}}^{M} A_i\right] \geqslant \sum_{\substack{i=1 \\ i \neq k}}^{M} P[A_i] - \sum_{\substack{i=2 \\ i \neq k}}^{M} \sum_{j=1}^{i-1} P[A_i A_j]$$

However, noting that $r_i\sqrt{2/N_0}$ is a unit variance Gaussian random variable with mean $\sqrt{2E_s/N_0}\,\delta_{ik}$, we have

$$P[A_i | r_k\sqrt{2/N_0} = y] = Q(y)$$

and

$$P[A_i] = \overline{Q(y)}$$

Similarly,

$$P[A_i A_j] = \overline{Q^2(y)}, \quad i \neq j$$

Thus

$$P[\mathcal{E}] \geqslant (M-1)\,\overline{Q(y)} - \frac{(M-1)(M-2)}{2}\,\overline{Q^2(y)}$$

This result may also be obtained by noting

5.9 continued

$$P[\mathcal{E}] = \overline{1 - [1 - Q(y)]^{M-1}} \geqslant (M-1)\overline{Q(y)} + \binom{M-1}{2}\overline{Q^2(y)}$$

However, $\overline{Q(y)}$ is the error probability for binary orthogonal signals, so

$$\overline{Q(y)} = Q\left(\sqrt{\frac{E_s}{N_0}}\right) \geqslant \frac{e^{-E_s/2N_0}}{\sqrt{2\pi E_s/N_0}}\left(1 - \frac{N_0}{E_s}\right)$$

c) From Appendix 7c, and the bound $Q(y) < e^{-y^2/2}$, we have

$$\overline{Q^2(y)} < \overline{e^{-y^2}} = \frac{\exp\{-\frac{2E_s}{N_0}/(1+2)\}}{\sqrt{1+2}} < e^{-\frac{2}{3}\frac{E_s}{N_0}}$$

Then, using $M = 2^{RT}$, $E_s = P_s T = C_\infty N_0 \ln 2\, T$

$$P[\mathcal{E}] \geqslant (M-1)e^{-\frac{E_s}{2N_0}}\left[\frac{1 - N_0/E_s}{\sqrt{2\pi E_s/N_0}} - \frac{M-2}{2}e^{-\frac{1}{6}\frac{E_s}{N_0}}\right]$$

$$> (1 - 2^{-RT})2^{-T(\frac{C_\infty}{2} - R)}\left[\frac{1 - 1/C_\infty T \ln 2}{\sqrt{2\pi C_\infty T \ln 2}} - \frac{1}{2}2^{-T(\frac{C_\infty}{6} - R)}\right]$$

$$= B\, 2^{-T(\frac{C_\infty}{2} - R)}$$

where B behaves like $1/\sqrt{T}$ with increasing T for $R \leqslant C_\infty/6$.

6.1

a)

x_3	x_2	x_1	y_5	y_4	y_3	y_2	y_1
0	1	1	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	0	0	1	1	0

we know for a parity check codes

$$\underset{\sim}{y} = \sum x_i \underset{\sim}{f}_i \quad \text{where additions are modulo -2}$$

Clearly $\underset{\sim}{f}_2 = (0\ 1\ 1\ 0\ 0)$ (elements from 1 to 5)

Note that a parity check coder is linear:

i) If we add the first entry to the last entry of the table we get

0 0 1 1 0 0 1 1

so

$$\underset{\sim}{f}_1 = (1\ 1\ 0\ 0\ 1)$$

ii) If we add all three table entries, we get

1 0 0 1 1 0 0 1

so

$$\underset{\sim}{f}_3 = 1\ 0\ 0\ 1\ 1$$

Connections are therefore