4.17 continued

$$(\underline{s}_i-\underline{a})\cdot(\underline{s}_j-\underline{a}) = \underline{s}_i\cdot\underline{s}_j - \underline{a}\cdot\underline{s}_i - \underline{a}\cdot\underline{s}_j + |\underline{a}|^2$$

$$= E_\rho\left\{[\rho+(1-\rho)\delta_{ij}] - \frac{2}{M}[1+(M-1)\rho] + \frac{1}{M^2}[M+M(M-1)\rho]\right\}$$

$$= E_\rho\left\{(1-\rho)\delta_{ij}+\rho\left(1 - \frac{2(M-1)}{M} + \frac{M-1}{M}\right) - \frac{2}{M}+\frac{1}{M}\right\}$$

$$= E_\rho\left\{(1-\rho)\delta_{ij} - \frac{1}{M}(1-\rho)\right\}$$

$$= \begin{cases} E_s\ , & i = j \\ -\frac{E_s}{M-1}\ , & i \neq j \end{cases}$$

Thus, the translated set is a simplex with energy $E_s = E_\rho(1 - \frac{1}{M})(1 - \rho)$ and hence both the original and the translated set have the same error probability as an M orthogonal signal set with energy $E_o = E_\rho(1 - \rho)$. (See p. 261 with $E_o$ substituted for $E_s$ in Eq. 499.) Note that as expected $\rho = 0 \Rightarrow E_\rho = E_o$, and $\rho = -\frac{1}{M-1} \Rightarrow E_\rho = E_s$.



5.2

From Eq. 5.15a,

$$P[\mathcal{E}] < \exp\left[-K\left(\frac{E_b}{2N_o} - \ln 2\right)\right]$$

$$= 2^{-\left[K\left(\frac{E_b}{E_{min}} - 1\right)\right]} = M^{-\left[\frac{E_b}{E_{min}} - 1\right]}$$

a) We desire $P[\mathcal{E}] < 10^{-6}$. From this bound,

i) $\frac{E_b}{E_{min}} = 1 \text{ db} = 1.26$

so $M^{-.26} = 10^{-6}$ or $M = 10^{6/.26} \approx 10^{24}$ (80 bits)

ii) $\frac{E_b}{E_{min}} = 3 \text{ db} = 2$

so $M^{-1} = 10^{-6}$ or $M = 10^6$ (20 bits)

iii) $\frac{E_b}{E_{min}} = 6 \text{ db} = 4$

so $M^{-3} = 10^{-6}$ or $M = 100$ ( 7 bits)

b) $K = R\ T$

$R = \frac{1}{.01} = 100$ bits/second

We must, therefore, wait T seconds to build up M messages, where

i) $T = \frac{K}{R} = \frac{80}{100} = 800$ msec

ii) $\frac{20}{100} = 200$ msec

iii) $\frac{7}{100} = 70$ msec

The use of M orthogonal signals in time T says we must provide M dimensions in time T. Thus, D must exceed M/T

$$D = \frac{3}{2} W > M/T$$

so

5.2 continued

$$W > \frac{2}{3}\frac{M}{T}$$

i) $W > \frac{2}{3}\frac{10^{24}}{.80} \approx 10^{24}$ cps

ii) $W > \frac{2}{3}\frac{10^{6}}{.2} \approx 3.3 \times 10^{6}$ cps

iii) $W > \frac{2}{3}\frac{128}{.07} \approx 1.2 \times 10^{3}$ cps

Note: the bound of Eq. 5.15a is useful to gain insight into bandwidth requirements. The bound of Eq. 5.106 is tighter however. It may be rewritten ($E_{b,min} \triangleq \ln 2$)

$$P[\epsilon] < \begin{cases} M^{-\left(\sqrt{\frac{E_b}{E_{b,min}}} - 1\right)^2} & 1 \leqslant \frac{E_b}{E_{b,min}} \leqslant 4 \\ M^{-\left(\frac{E_b}{2E_{b,min}} - 1\right)} & 4 < \frac{E_b}{E_{b,min}} \end{cases}$$



5.3

a) A set of M simplex signals with energy $E_s$ has the same error probability as a set of M orthogonal signals with energy $E_s \frac{M}{M-1}$. Thus

$$P(\mathcal{E}) \leqslant (M-1)Q\left(\sqrt{\frac{E_s M}{N_o(M-1)}}\right)$$

$$\leqslant M e^{-\frac{E_s M}{2N_o(M-1)}}$$

now

$$E_s = P_s T \text{ , and } M = 2^{RT} = e^{RT \ln 2}$$

$$P[\mathcal{E}]_{simplex} \leqslant e^{-T\left[\frac{P_s}{2N_o}\frac{M}{M-1} - R \ln 2\right]} .$$

b) For M biorthogonal signals, we again have, by the symmetry:

$$P[\mathcal{E}]_{biorthogonal} = P[\mathcal{E}|m_1] .$$

The same union bound holds, as always. Two different $P_2$ values come into consideration, however. The distance from $s_1$ to every other signal orthogonal to it is obviously

$$d^2_{ort} = 2E$$

There is one signal for which $s_k = -s_1$ and in this case

$$d^2_{ant} = 4E$$

So

$$P[\mathcal{E}|m_1] \leqslant \sum_{\substack{k=0 \\ i \neq k}}^{M-1} P_2[s_i, s_k] = Q\left(\sqrt{\frac{4E}{2N_o}}\right) + (M-2)Q\left(\sqrt{\frac{2E}{2N_o}}\right)$$

5.3 continued

$$< Q\left(\sqrt{\frac{2E}{2N_o}}\right) + (M-2)Q\left(\sqrt{\frac{2E}{2N_o}}\right) = (M-1)Q\left(\sqrt{\frac{2E}{2N_o}}\right)$$

$$< M\, Q\left(\sqrt{\frac{2E}{2N_o}}\right)$$

$$< e^{RT \ln 2}\, e^{-E/2N_o}$$

$$= e^{RT \ln 2 - \frac{PT}{2N_o}} = e^{-T\left[\frac{P}{2N_o} - R \ln 2\right]}$$

These all have roughly the same error behavior for large M. There are good reasons for the use of each;

(1) Orthogonal signals are easily generated (for instance sine wave pulse of different frequencies).

(2) Simplex signals can be generated easily for binary data by a maximal length shift register.

(3) Biorthogonal signals use only half the bandwidth required by the other two presented here.



5.4

First, let's apply the bandwidth constraint. To do so we need the spectrum. This is

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2/2\sigma^2}\, e^{-j2\pi f t} dt$$

Recalling that $E(e^{j\nu x}) = e^{-1/2(\sigma^2\nu^2)}$ for a zero mean Gaussian r.v.x we obtain, with $\nu = 2\pi f$,

$$X(f) = e^{-1/2(\sigma^2)(2\pi f)^2} = e^{-2\pi^2\sigma^2 f^2}$$

The shifted pulses add only a linear phase factor but do not affect bandwidth.

Note

$$|X(f)|^2 = e^{-(2\pi\sigma)^2 f^2} = e^{-\left(\frac{f^2}{2(1/8\pi^2\sigma^2)}\right)} = e^{-f^2/2\Sigma^2}$$

We desire

$$\frac{\int_{-W}^{W} |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df} > .9$$

Note

$$\frac{|X(f)|^2}{\int_{-\infty}^{\infty} |X(f)|^2 df}$$ is a normalized Gaussian density function

So from a table of $Q(\alpha)$ we find

$$W \geqslant 1.645\,\Sigma = \frac{1.645}{2\pi\sigma\sqrt{2}}$$

That is, since W is fixed (or selected),

$$\sigma \geqslant \frac{1.645}{2\pi\sqrt{2}\,W} \approx .185 \frac{1}{W}$$

Now that we've selected some permissible $\sigma$, let's look for a $\sigma$ which meets the overlap condition.

5.4 continued

$$\int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt \le .05 \int_{-\infty}^{\infty} x(t)\, x(t)\, dt$$

Clearly larger (say $2\tau$) delays cause less overlap.

However, both integrals can be obtained as the convolution of 2 zero-mean, variance $\sigma^2$ Gaussian density functions, one evaluated at $\tau$ and the other at 0. The result is a Gaussian density function of zero mean and variance $2\sigma^2$. Hence, the condition on $\tau$ is

$$\frac{1}{\sqrt{2\pi\, 2\sigma^2}}\, e^{-\tau^2/4\sigma^2} \le .05\, \frac{1}{\sqrt{2\pi\, 2\sigma^2}}\, e^{-0}$$

or

$$\tau^2/4\sigma^2 > -\ln\ (.05)$$

$$\tau > 2\sigma\sqrt{-\ln\ (.05)}$$

$$\approx 3.47\,\sigma$$

The number of x(t) translates that fit the conditions above and which can be put into T seconds is the integer just below $T/\tau$. As T gets very large, we can say it equals $T/\tau$ without making much of an error, percentage-wise.

Thus, the number of dimensions available using these



5.4 continued

pseudo-orthonormal signals is

$$N \sim T/\tau < \frac{T}{3.47\sigma} < \frac{T}{3.47 \times .185} \frac{W}{} = 1.57\ TW$$

that is

$$k \approx 1.57$$

5.6

c) $$\overline{E_N} = \frac{JP_s\tau}{N} = \begin{cases} P_s\tau & \text{for } A = 2 \\ \frac{P_s\tau}{2} & \text{for } A \geq 3 \end{cases}$$

Then

$$\frac{\overline{E_N}}{N_o} \to 0$$

is equivalent to

$$\frac{P_s\tau}{N_o} \to 0 \quad \text{for } A = 2 \qquad , \qquad \frac{P_s\tau}{2N_o} \to 0 \quad \text{for } A \geq 3$$

We need not consider A = 2 separately, since it is identical to A = 4 for the same value of $\overline{E_N}$.

Let $X = P_s\tau/N_o = 2E_N$ for $A \geq 3$. Then

$$\lim_{x\to 0}\left[-\frac{1}{2}\log_2\left(\frac{1}{A}\sum_{\ell=1}^{A}\exp\left[-x\sin^2 \ell \frac{\pi}{A}\right]\right)\right]$$

$$=\lim_{x\to 0}\left[-\frac{1}{2\ln 2}\ln\left(\frac{1}{A}\sum_{\ell=1}^{A}\left(1 - x\sin^2 \ell \frac{\pi}{A}\right)\right)\right]$$

$$=\lim_{x\to 0}\left[-\frac{1}{2\ln 2}\ln\left(1 - \frac{x}{A}\left(\sum_{\ell=1}^{A-1}\sin^2 \ell \frac{\pi}{A} + 0\right)\right)\right]$$

$$=\lim_{x\to 0}\left[-\frac{1}{2\ln 2}\ln\left(1 - \frac{x}{2}\right)\right]$$

$$= \frac{x}{4\ln 2} = \frac{\overline{E_N}}{N_o}\frac{1}{2\ln 2}$$

Thus, for very low rates, A-ary phase modulated coded signals are as efficient as binary antipodal coded signals. Both can achieve an arbitrarily low error probability if $E_b/N_o > 2\ln 2$ and $\overline{E_N}/N_o$ is sufficiently small. For phase-modulated signals, and in contrast with amplitude-modulated signals for A > 2, all M possi-



5.6 continued

ble waveforms have energy $N\,\overline{E_N}$.

5.6

$$\psi_k = \sqrt{\frac{2}{\tau}} \sin 2\pi\left(\frac{Lt}{\tau} + \frac{k}{A}\right), \quad -\tau \le t < 0, \quad k = 1,2,\ldots,A$$

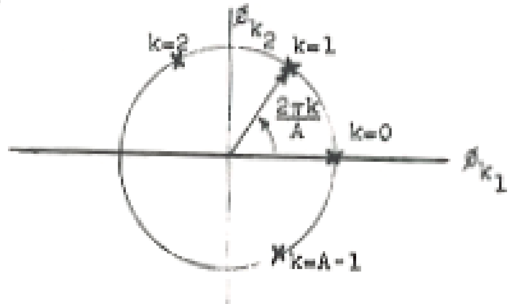
These $\psi_k(t)$'s can be expanded in two orthonormal functions;

$$\phi_{k_1}(t) = \sqrt{\frac{2}{\tau}} \sin 2\pi \frac{Lt}{\tau} \qquad -\tau \leqslant t < 0$$

and

$$\phi_{k_2}(t) = \sqrt{\frac{2}{\tau}} \cos 2\pi \frac{Lt}{\tau} \qquad -\tau \leqslant t < 0$$

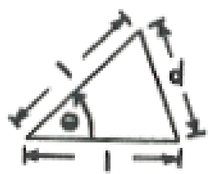
for example:




Note that for A = 2, this reduces to two binary antipodal signals. From the union bound, Eq. 5.29,

$$\overline{P[\mathcal{E}|m_k]} < \sum_{\substack{i=0\\ i\neq k}}^{M-1} P_2[\underset{\sim}{S}_i, \underset{\sim}{S}_k]$$

For the white Gaussian noise channel,




$d = 2 \sin \frac{\theta}{2}$

(a)

$$P_2[\underset{\sim}{S}_i, \underset{\sim}{S}_k] = Q\left(\frac{|\underset{\sim}{S}_i - \underset{\sim}{S}_k|}{\sqrt{2N_0}}\right)$$

$$< e^{-|\underset{\sim}{S}_i - \underset{\sim}{S}_k|^2/4N_0}$$

If $\underset{\sim}{S}_i$ is specified by the vector $(k_{i_1}, k_{i_2}, \ldots, k_{i_J})$ and similarly for $\underset{\sim}{S}_k$, then from (a)

$$|\underset{\sim}{S}_i - \underset{\sim}{S}_k|^2 = P_s\tau \sum_{j=1}^{J} 4 \sin^2 \frac{\pi}{A}(k_{i_j} - k_{k_j})$$



5.6 continued

Over the ensemble of codes, $\frac{\pi}{A}|k_{i_j} - k_{k_j}|$ is equally likely to have any of the values $0, \frac{\pi}{A}, \ldots, \frac{\pi(A-1)}{A}$ independently for each j, j=1,...,J. Hence

$$\overline{P_2[S_i, S_k]} < \sum_{j=1}^{J} \exp\left\{-\frac{P_s\tau}{N_0} \sin^2 \frac{\pi}{A}(k_{i_j} - k_{k_j})\right\}$$

$$= \prod_{j=1}^{N} \frac{1}{A} \sum_{\ell=1}^{A} \exp\left\{-\frac{P_s\tau}{N_0} \sin^2 \frac{\ell\pi}{A}\right\}$$

$$= 2^{-NR_0}$$

in which (N = 2J for J ⩾ 3; N = J for J = 2)

$$R_0 = -\alpha \log_2 \frac{1}{A} \sum_{\ell=1}^{A} \exp\left\{-\frac{P_s\tau}{N_0} \sin^2 \ell \frac{\pi}{A}\right\}$$

$$\alpha = \begin{cases} 1 & , A = 2 \\ 1/2 & , A \geqslant 3 \end{cases}$$

Then,

$$\overline{P[\mathcal{E}]} = \overline{P[\mathcal{E}|m_k]} < M\, 2^{-NR_0} = 2^{-N(R_0 - R_N)}$$

b) For A = 2, we have the case of binary antipodal signaling. For A = 4, a code word consists of N/2 components from a four-letter biorthogonal alphabet (4 phases differing by 90°) which can be viewed as N components from a binary antipodal alphabet. Hence, $R_0$ for A = 2 must equal $R_0$ for A = 4. For A = 4,

$$R_0 = -\frac{1}{2} \log_2 \frac{1}{4}\left[1 + 2e^{-P_s\tau/2N_0} + e^{-P_s\tau/N_0}\right]$$

$$= -\log_2 \frac{1}{2}\left[1 + e^{-P_s\tau/2N_0}\right] = -\log_2 \frac{1}{2}\left[1 + e^{-E_N/N_0}\right]$$

Q.E.D.