3.18 continued

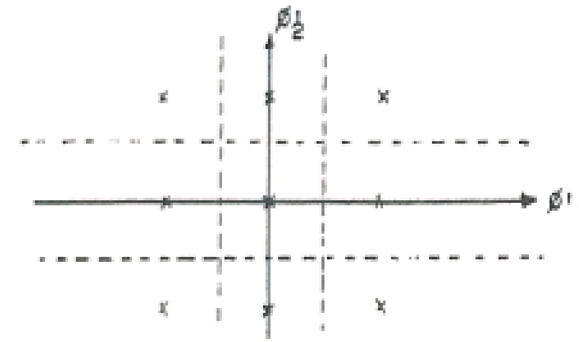
$$y = \sum_{i=1}^{N} a_i y_i = \sum_{j=1}^{K} \left(\sum_{i=1}^{N} a_i b_{ij} \right) x_j$$

is a weighted linear sum of the x_i's and hence is Gaussian.

4.3

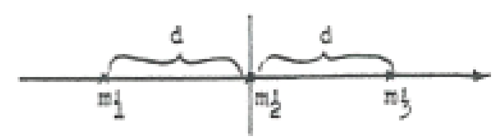
White Gaussian noise is spherically symmetric, so we may rotate and translate the signal constellation to any plane we wish without affecting error performance. That is to say, / and θ do not affect the result of this problem at all.

So, let's look at this signal set after rotation and translation.



Optimum decision boundaries are indicated by the broken lines.

Due to the decision region geometry, we can say that in order to be correct, we must be correct in both the ϕ'_1 and ϕ'_2 directions. Furthermore, because the messages are equally likely and the noise components are statistically independent, there is no loss in optimality (by the theorem of irrelevance) of making separate decisions on each dimension. (See p. 257.)



4.2 continued

Thus

$$P[C] = P[C \text{ on } \phi_1^i] \; P[C \text{ on } \phi_2^i]$$

$$= (1 - \tfrac{4}{3} q) \quad (1 - \tfrac{4}{3} q)$$

where

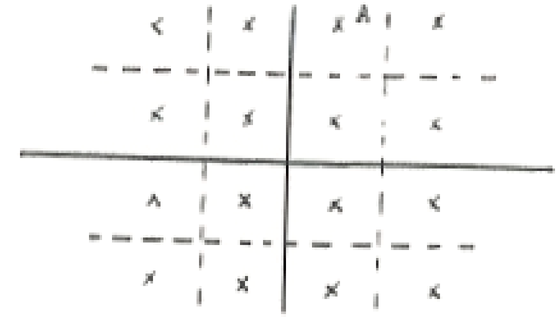
$$1 - \tfrac{4}{3} q = \sum_{i=1}^{3} P[C|m_i] \, P[m_i] = \tfrac{1}{3}[(1-q) + (1-2q) + (1-q)]$$

Therefore

$$P(\epsilon) = 1 - (1 - \tfrac{4}{3} q)^2 = \tfrac{8}{9} q [3 - 2q]$$

4.4

a) Optimum decision boundaries are perpendicular bisectors of lines between signal points. Thus:



The axes are also boundaries.

b) Due to the geometry of the decision regions (see top p. 257) we can say

$$P[C] = P[C \text{ on } \phi_1 \text{ and } C \text{ on } \phi_2]$$

$$= P[C \text{ on } \phi_1] \cdot P[C \text{ on } \phi_2]$$

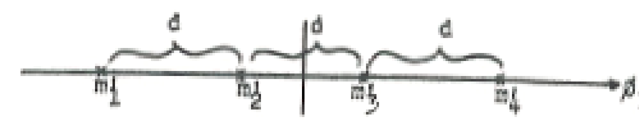
$$= (1 - \tfrac{6}{4} q) \; (1 - \tfrac{6}{4} q)$$

where

$$q = Q\left(\frac{d}{\sqrt{2 N_0}}\right)$$

and where

$$(1 - \tfrac{6}{4} q) = \tfrac{1}{4} \sum_{i=1}^{4} P[C|m_i] = \tfrac{1}{4}[(1-q) + (1-2q) + (1-2q) + (1-q)]$$



$$P(\epsilon) = 1 - P(C) = 1 - (1 - \tfrac{6}{4} q)^2 = 3q - \tfrac{9}{4} q^2$$

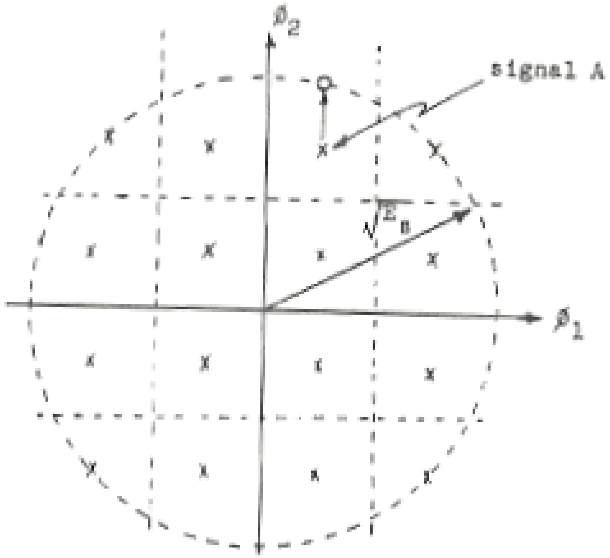
c) For the signals of Fig. P4.4, the maximum signal energy is

4.4 continued

$$E_s = (\tfrac{3}{2} d)^2 + (\tfrac{3}{2} d)^2 = \tfrac{9}{2} d^2$$

(the corner signals have energy E_s). We can improve error behavior as follows.

Keep the decision boundaries fixed, but move signal A upward along ϕ_2 keeping the ϕ_1 component fixed until it has energy E_s. Note that this does not affect the error performance of any other signal, since we have kept the boundaries fixed. It does, however, decrease the error probability on signal A, and this improves the over-all error probability. Of course, the optimum receiver for this modified signal set would do even better.



4.6

a) The optimum decision is to select the value of i which maximizes

$$p_{r_1,r_2}(\rho_1,\rho_2|m_i) = p_{r_1}(\rho_1|m_i)p_{r_2}(\rho_2|m_i,r_1=\rho_1)$$

$$= p_{n_1}(\rho_1 - s_i)p_{n_2}(\rho_2 - [\rho_1 - s_i])$$

$$= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left[(\rho_1-s_i)^2+(\rho_2-\rho_1+s_i)^2\right]\right\}$$

or equivalently, to minimize

$$\rho_1^2 - 2\rho_1 s_i + s_i^2 + (\rho_2 - \rho_1)^2 + 2(\rho_2 - \rho_1)s_i + s_i^2$$

Noting $s_i^2 = E_s$, $i = 0,1$, we discard all terms independent of i and <u>maximize</u>

$$4(\rho_1 - \tfrac{1}{2}\rho_2)s_i$$

b) The optimum value of a is $(-\tfrac{1}{2})$.

c) $\hat{m} = m_0$ if and only if $r_1 - \tfrac{1}{2} r_2 > 0$. Thus, the threshold setting is zero.

The above answers may also be obtained by use of the theorem on irrelevance. First note that $r \triangleq r_1 - \tfrac{1}{2} r_2$ and r_2 can be used as input to the receiver without loss of optimality, since r_1 can always be recovered. However,

$$r = r_1 - \tfrac{1}{2} r_2 = s + \eta$$

where

$$\eta = \tfrac{1}{2}(n_1 - n_2)$$

is statistically independent of $r_2 = n_1 + n_2$:

$$\text{cov}(\eta, n_1 + n_2) = \overline{\eta(n_1 + n_2)} = \tfrac{1}{2}(\overline{n_1^2} - \overline{n_2^2}) = 0$$

Thus, since r_2 contains no signal component, it is

4.6 continued

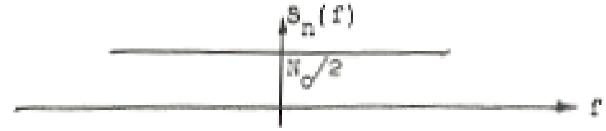
irrelevant to the decision when r is available. Since $r = s + \xi$, $s = \pm\sqrt{E_s}$, and $\sigma_\xi^2 = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{1}{2}\sigma^2$, we have

d) $$P[E] = Q\left(\sqrt{\frac{2E_s}{\sigma^2}}\right)$$

e) Since $r_1 = s + n_1$ and $\sigma_{n_1}^2 = \sigma^2$, $P[\mathcal{E}] = Q\left(\sqrt{\frac{E_s}{\sigma^2}}\right)$. If only r_1 is available, E_s must be increased by 3db to obtain the same error probability.

4.7

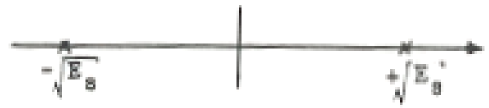
We are told that $x(t)$ and $y(t)$ are equi-energy, orthogonal signals, with energy $E_s = 16$ joules. Noise power density $N_o/2 = 4$ watts/cps (double ended spectrum)



a) for case 1,

$$S_1 = x(t) = \sqrt{E_s}\,\phi_1(t)$$
$$S_2 = -x(t) = -\sqrt{E_s}\,\phi(t)$$

the signal constellation is



for a white Gaussian noise channel

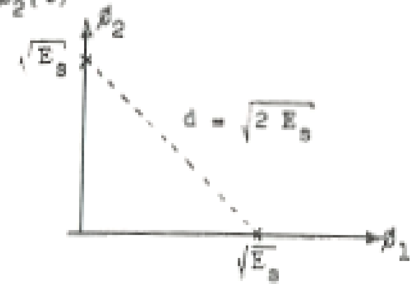
$$P[\epsilon] = Q\left(\frac{d}{\sqrt{2N_o}}\right) \quad \text{where } d = 2\sqrt{E_s}$$

$$= Q\left(\frac{2\sqrt{16}}{\sqrt{2\cdot 8}}\right) = Q(2) \approx .023$$

b) for case 2,

$$S_1 = x(t) = \sqrt{E_s}\,\phi_1(t)$$
$$S_2 = y(t) = \sqrt{E_s}\,\phi_2(t)$$

the signal constellation is



in this case $P[\epsilon] = Q\left(\frac{\sqrt{2.16}}{\sqrt{2.8}}\right) = Q(\sqrt{2}) \approx .08$

4.10

a) An optimum receiver selects m_o after receiving $r(t)$ if and only if

$$\int_{-\infty}^{\infty} r(t)s_o(t)dt > \int_{-\infty}^{\infty} r(t)s_1(t)dt$$

since the energy in $s_o(t)$ equals that in $s_1(t)$.

b) For equally-likely binary messages,

$$P[\epsilon] = Q\left(\frac{d}{\sqrt{2N_o}}\right)$$

where

$$d^2 = |\underline{s}_o - \underline{s}_1|^2$$

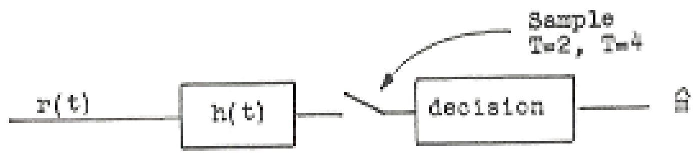
$$= \int_{-\infty}^{\infty} [s_o(t) - s_1(t)]^2 dt$$

$$= 2\int_{-\infty}^{\infty} p^2(t)dt = 4\int_0^1 (2x)^2 dx = \frac{16}{3}$$

Thus,

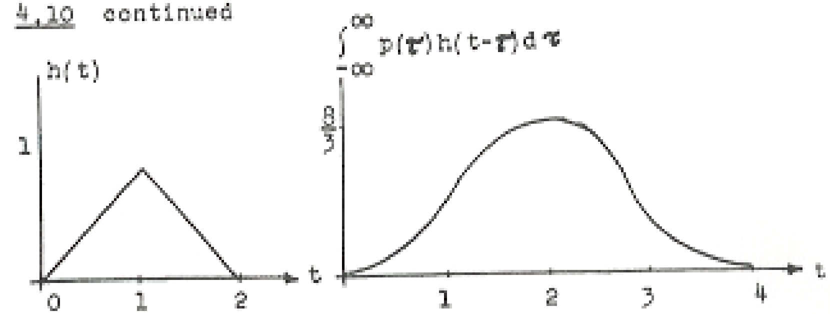
$$P[\epsilon] = Q\left(\frac{4}{\sqrt{6x.1}}\right) = Q\left(\frac{4}{\sqrt{.6}}\right) \approx 10^{-7}$$

c) Method I

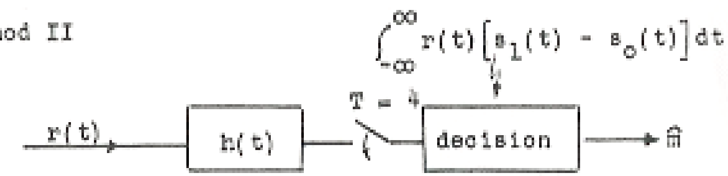


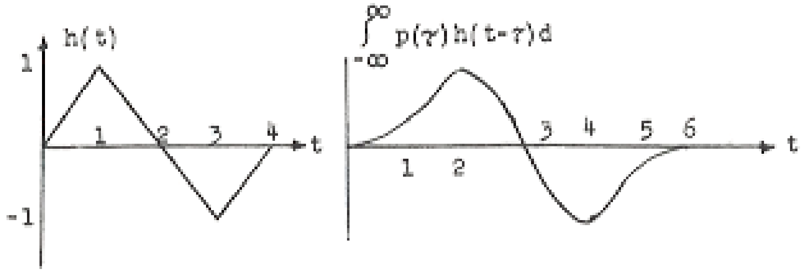
The sample at $T = 2$ is $\int_{-\infty}^{\infty} r(t)s_o(t)dt$. The sample at $T = 4$ is $\int_{-\infty}^{\infty} r(t)s_1(t)dt$.

4.10 continued



Method II





Method I is most easily generalized in that Method II requires a filter for each s_i-s_j pair.

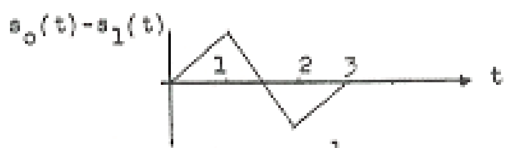
d) Same error probability: $s_o(t) = \frac{1}{2}[p(t) + p(t-2)]$

$s_1(t) = \frac{1}{2}[p(t) - p(t-2)]$

Lower error probability: $-s_o(t) = s_1(t) = p(t)$

e) For $s_o(t) = p(t)$; $s_1(t) = p(t-1)$

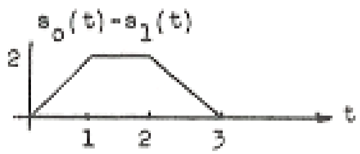
4.10 continued



$$d^2 = \int_{-\infty}^{\infty}[s_o(t) - s_1(t)]^2 dt = 2\int_0^1 (2x)^2 dx + 2\int_0^{\frac{1}{2}} (4x)^2 dx$$

$$= 2\left(\frac{4}{3} + \frac{16}{3.8}\right) = 4$$

$$P(\epsilon) = Q\left(\frac{1}{\sqrt{.05}}\right) \approx 4 \times 10^{-6}$$



For $s_o(t) = p(t)$; $s_1(t) = -p(t-1)$

$$d^2 = \int_{-\infty}^{\infty} [s_o(t) - s_1(t)]^2 dt = 2\int_2^1 (2x)^2 dx + 4 = \frac{20}{3}$$

$$P(\epsilon) = Q\left(\sqrt{\frac{5}{3(.05)}}\right) = Q\left(\frac{10}{\sqrt{3}}\right) \approx 4 \times 10^{-9}$$

4.12

(a) Let $s_i(t) = a(t)$, $t_1 \le t \le t_2$, $i = 0, 1, \ldots M-1$.

The optimum receiver for white Gaussian noise bases its decision on

$$\int_{-\infty}^{\infty} r(t)\, s_i(t) dt + c_i \qquad \text{(Eq. 4.60)}$$

$$= \int_{t_1}^{t_2} r(t)\, a(t)\, dt + \int_{t \notin [t_1, t_2]} r(t)\, s_i(t) dt + c_i$$

Since the integral over $[t_1, t_2]$ is independent of i, the optimum receiver may ignore this interval and hence ignore r(t) for t in $[t_1, t_2]$.

A similar argument holds in the vector case. We might reduce the waveform case to the vector case, of course, by choosing $a(t) / \sqrt{\int a^2(t) dt}$ as an orthonormal function which is zero outside of $[t_1, t_2]$. All signals would then have the same projection in this dimension.

(b) No. As an extreme example, consider a case in which $n(t) = n$, all t, and n is a Gaussian random variable. If the signals are $s_o(t) = 1$, $0 < t < 1$

$$s_1(t) = \begin{cases} 1 & 0 < t \le \frac{1}{2} \\ -1 & \frac{1}{2} < t < 1 \end{cases}$$

then a perfect decision is possible if r(t) is examined in the interval $[0, \frac{1}{2}]$ as well as in $[\frac{1}{2}, 1]$ whereas a non-zero error probability results if r(t) is examined only for $\frac{1}{2} < t < 1$.

4.15

Assuming $S_N(f) = N_o/2$ we have

$$P[\epsilon] = Q\left(\frac{d}{\sqrt{2N_o}}\right)$$

Truncated r(t):

$$d^2 = \int_0^2 [s_o(t) - s_1(t)]^2 dt = 4\int_0^2 e^{-2t}dt = 2(1 - e^{-4})$$

all of r(t):

$$d^2 = 4\int_0^{\infty} e^{-2t}dt = 2$$

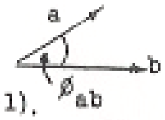
We never lose much in error performance, as the following table shows:

N_o	P[ε] truncated	P[ε] complete	P[ε] truncated / P[ε] complete
50	.4443	.4438	1.001
.5	.0806	.0787	1.025
.05	$.468 \times 10^{-5}$	$.387 \times 10^{-5}$	1.211
.02	$.123 \times 10^{-11}$	$.774 \times 10^{-12}$	1.595

4.17

Given $\underset{\sim}{\theta}_i \cdot \underset{\sim}{\theta}_j = \delta_{ij} + (1 - \delta_{ij})\rho$

a) remember $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos \phi_{ab}$



so $\rho = \underset{\sim}{\theta}_i \cdot \underset{\sim}{\theta}_j |_{i \neq j} = |\theta_i||\theta_j| \cos \theta_{ij}$

$= \cos \phi_{ij} \leqslant 1$ (and $\geqslant -1$).

Both lower and upper bounds may be obtained from the observation

$$M^2 = \left(\sum_{i=0}^{M-1} |\underset{\sim}{\theta}_i|\right)^2 \geqslant \left|\sum_{i=0}^{M-1} \underset{\sim}{\theta}_i\right|^2 = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \underset{\sim}{\theta}_i \cdot \underset{\sim}{\theta}_j$$

$$= \sum_{i=0}^{M-1} \underset{\sim}{\theta}_i \cdot \underset{\sim}{\theta}_j + \sum_{i=0}^{M-1} \sum_{\substack{j=0 \\ i \neq j}}^{M-1} \underset{\sim}{\theta}_i \cdot \underset{\sim}{\theta}_j$$

$= M(1) + (M^2 - M)\rho \geqslant 0$ (it's the magnitude square of a vector)

$$M \geqslant 1 + (M - 1)\rho \geqslant 0 \qquad \underline{1 \geqslant \rho \geqslant -\frac{1}{M-1}}$$

We must still show equality for simplex using 4.99 with $E_s(1 - \frac{1}{M}) = 1$ (unit energy simplex).

Then $E_s = \frac{M}{M-1}$ and

$$\underset{\sim}{\theta}_i \cdot \underset{\sim}{\theta}_j = \begin{cases} 1 & i = j \\ -\frac{1}{M-1} & i \neq j \end{cases}$$

if $\{\theta_i\}$ constitutes a unit energy simplex.

b) The signal sets $\{\underset{\sim}{s}_i - \underset{\sim}{s}\}$ and $\{\underset{\sim}{s}_i\}$ differ only by a translation and hence have the same error probability.

Let $\underset{\sim}{s} = \frac{1}{M}\sum_{i=0}^{M-1} \underset{\sim}{s}_i$. Thus

4.17 continued

$$(\mathbf{s}_i-\mathbf{s})\cdot(\mathbf{s}_j-\mathbf{s}) = \mathbf{s}_i\cdot\mathbf{s}_j - \mathbf{s}\cdot\mathbf{s}_i - \mathbf{s}\cdot\mathbf{s}_j + |\mathbf{s}|^2$$

$$= E_\rho\left\{[\rho+(1-\rho)\delta_{ij}] - \frac{2}{M}[1+(M-1)\rho] + \frac{1}{M^2}[M+M(M-1)\rho]\right\}$$

$$= E_\rho\left\{(1-\rho)\delta_{ij} + \rho\left(1 - \frac{2(M-1)}{M} + \frac{M-1}{M}\right) - \frac{2}{M} + \frac{1}{M}\right\}$$

$$= E_\rho\left\{(1-\rho)\delta_{ij} - \frac{1}{M}(1-\rho)\right\}$$

$$= \begin{cases} E_s, & i = j \\ -\dfrac{E_s}{M-1}, & i \neq j \end{cases}$$

Thus, the translated set is a simplex with energy $E_s = E_\rho(1 - \frac{1}{M})(1 - \rho)$ and hence both the original and the translated set have the same error probability as an M orthogonal signal set with energy $E_o = E_\rho(1 - \rho)$. (See p. 261 with E_o substituted for E_s in Eq. 499.) Note that as expected $\rho = 0 \Rightarrow E_\rho = E_o$, and $\rho = -\frac{1}{M-1} \Rightarrow E_\rho = E_s$.

5.2

From Eq. 5.15a,

$$P[\mathcal{E}] < \exp\left[-K\left(\frac{E_b}{2N_o} - \ln 2\right)\right]$$

$$= 2^{-\left[K\left(\frac{E_b}{E_{min}} - 1\right)\right]} = M^{-\left[\frac{E_b}{E_{min}} - 1\right]}$$

a) We desire $P[\mathcal{E}] < 10^{-6}$. From this bound,

i) $\frac{E_b}{E_{min}} = 1$ db $= 1.26$

so $M^{-.26} = 10^{-6}$ or $M = 10^{6/.26} \approx 10^{24}$ (80 bits)

ii) $\frac{E_b}{E_{min}} = 3$ db $= 2$

so $M^{-1} = 10^{-6}$ or $M = 10^6$ (20 bits)

iii) $\frac{E_b}{E_{min}} = 6$ db $= 4$

so $M^{-3} = 10^{-6}$ or $M = 100$ (7 bits)

b) $K = R\,T$

$R = \frac{1}{.01} = 100$ bits/second

We must, therefore, wait T seconds to build up M messages, where

i) $T = \frac{K}{R} = \frac{80}{100} = 800$ msec

ii) $\frac{20}{100} = 200$ msec

iii) $\frac{7}{100} = 70$ msec

The use of M orthogonal signals in time T says we must provide M dimensions in time T. Thus, D must exceed M/T

$$D = \frac{3}{2} W > M/T$$

so