p on n-th toss] = [not stop on 1st toss $\cap$ ---

$\cap$ not stop at (n-1)th toss $\cap$ stop on n-th toss]

= [face 2 $\neq$ face 1 $\cap$ --- $\cap$ Face (n-1) $\neq$ face (n-2) $\cap$ Face n = face (n-1)]

(2: number of throw)

Since the successive throws are statistically independent, we

'e

$$P[\text{stop on n-th toss}] = P[\text{face } 2 \neq \text{face } 1] \cdot \ldots \cdot P[\text{face } (n-1) \neq \text{face } (n-2)] \cdot P[\text{face } n = \text{face } (n-1)]$$

$$= \underbrace{\frac{5}{6} \cdot \ldots \cdot \frac{5}{6}}_{(n-2)\text{ times}} \cdot \frac{1}{6} = \underline{\underline{\left(\frac{5}{6}\right)^{n-2} \cdot \frac{1}{6}}}$$

$$\sum_{n=2}^{\infty} P[\text{stop on n-th toss}] = \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \ldots\right]$$

$$= \frac{1}{6} \cdot \frac{1}{1-5/6} = \frac{1}{6} \cdot \frac{1}{\frac{1}{6}} = 1 \quad \text{as expected.}$$

ote that in this example the sample space contains a (countably) finite number of points.

2.23

For equally likely messages, the maximum a posteriori probability receiver reduces to the maximum likelihood receiver. Hence, we have

$m = m_o$ if and only if $p_r(\rho|m_o) > p_r(\rho|m_1)$.

From the figure, we observe that

$m = m_o$ if and only if $\frac{1}{4} < \rho < \frac{7}{4}$.

The error probability is

$$P(\epsilon) = P[m_o]\left[1 - \int_{\frac{1}{4}}^{\frac{7}{4}} p_r(\rho|m_o)\,d\rho\right] + P[m_1]\int_{\frac{1}{4}}^{\frac{7}{4}} p_r(\rho|m_1)\,d\rho$$

$$= \frac{1}{2} \cdot \frac{1}{16} + \frac{1}{2} \cdot \frac{6}{16} = \frac{7}{32}.$$

2.24

The quantized version of Prob. 2.23 is mathematically simpler than the unquantized, since only discrete probability theory is required. Since the messages are equally likely, the maximum a posteriori probability receiver is the same as the maximum likelihood receiver. Let $q_{i,j}$, $i = 0, 1$; $j = -2, -1, \ldots, +3$; be the probability of receiving j when i is transmitted. A straightforward calculation yields

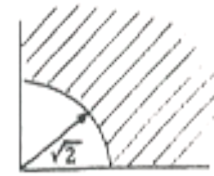
$q_{0,3} = 0$;	$q_{1,3} = 1/8$;	$m(3) = 1$
$q_{0,2} = 1/8$;	$q_{1,2} = 1/4$;	$m(2) = 1$
$q_{0,1} = 3/4$;	$q_{1,1} = 1/4$;	$m(1) = 0$
$q_{0,0} = 1/8$;	$q_{1,0} = 1/4$;	$m(0) = 1$
$q_{0,-1} = 0$;	$q_{1,-1} = 1/8$;	$m(-1) = 1$
$q_{0,-2} = 0$;	$q_{1,-2} = 0$;	$m(-2) = 1$

$$P[\epsilon] = P[m_0][1 - q_{0,1}] + P[m_1]q_{1,1} = 1/4$$

Note that quantization increases the error probability (above 7/32) as expected, although the percentage increase is small.

2.26

$$[Q(\alpha)]^2 = \left[\int_\alpha^\infty \sqrt{\frac{1}{2\pi}}\, e^{-\rho^2/2}\, d\rho\right]^2 \leqslant \int\int_R \frac{1}{2\pi} e^{-(u^2+v^2)/2} du\,dv$$



Let $\quad u^2 + v^2 = r^2$

$\quad du\,dv = r\,dr\,d\theta$

$$[Q(\alpha)]^2 \leq \int_0^{\pi/2} d\theta \int_{\sqrt{2}\,\alpha}^{\infty} \frac{1}{2\pi} e^{-r^2/2}\, r\,dr \qquad \alpha > 0$$

$$= \frac{1}{4}\left(-e^{-r^2/2}\right)\Big|_{\sqrt{2}\,\alpha}^{\infty}$$

$$= \frac{1}{4} e^{-\alpha^2}$$

so

$$Q(\alpha) \leq \frac{1}{2} e^{-\alpha^2/2} \quad , \quad \alpha \geq 0.$$

The equality holds only if $\alpha = 0$.

This bound is tighter than $\frac{1}{\sqrt{2\pi}\,\alpha} e^{-\alpha^2/2}$ whenever $\frac{1}{2} < \frac{1}{\sqrt{2\pi}\,\alpha}$; hence whenever $0 \leqslant \alpha < \sqrt{\frac{2}{\pi}} \approx 0.8$.

2.38

NOTE: The problem should state that the true channel error probability $P[\epsilon]$ may be assumed to be greater than 0.01. We may then assume $P[\epsilon] = p = 0.01$ to obtain a conservative estimate of the number of channel uses N.

We are interested in selecting N so that

$$P_1 = P\left[\left|\frac{1}{N}\sum_{i=1}^{N} x_i - p\right| \geq .05\,p\right] \leq .001$$

in which the $\{x_i\}$ are statistically independent, binary random variables with

$$P[x_i = 1] = p$$

$$P[x_i = 0] = q = 1 - p$$

Thus, $x_i = 1$ denotes an error on the i^{th} use of the channel. Note that

$$\bar{x}_i = p$$

$$\sigma_{x_i}^2 = p(1-p) = \sigma^2$$

Let $\epsilon = .05\,p$.

(a) Chebyshev inequality:

$$P[\,|y| \geq \epsilon] \leq \frac{\overline{y^2}}{\epsilon^2}$$

or

$$P_1 \leq \frac{\sigma^2/N}{\epsilon^2} = \frac{p(1-p)/N}{(.05p)^2} = .001$$

2.38 continued

$$N = \frac{1-p}{p}\,\frac{1}{25}\,10^7 \approx \underline{4 \times 10^7 \text{ Trials}}$$

(b) Central limit theorem

Assume that $y = \frac{1}{N}\sum_{i=1}^{N} x_i$ is Gaussian with mean p and variance σ^2/N. Then

$$P_1 \approx 2\,P[y \geq \epsilon] = 2\,Q\left(\frac{\epsilon}{\sqrt{\sigma^2/N}}\right) = 2\,Q\left(\frac{.05p}{\sqrt{p(1-p)/N}}\right) = .001.$$

Using an approx (Eq. 2.121) to $Q(\alpha)$ yields

$$P_1 \approx \frac{2}{\sqrt{2\pi}\,\epsilon\,\sqrt{\sigma^2/N}}\,\exp\left[-N\frac{\epsilon^2}{2\sigma^2}\right].$$

Using table lookup of the Q function, instead of the approximation

$$\frac{.05p}{\sqrt{p(1-p)/N}} = 3.9$$

and

$$\underline{N = 6.1 \times 10^5}.$$

(Note: We do not know whether or not this choice of N is conservative.)

(c) Chernoff bound

From Eq. 2.166, we find

$$P\left[\frac{1}{N}\sum_{i=1}^{N} x_i > \tau\right] \leq e^{-NX}, \quad \tau > p$$

2.38 continued

$$P\left[\frac{1}{N}\sum_{i=1}^{N} x_i < \tau\right] \le e^{-NX}, \quad \tau < p$$

in which

$$X = T_p(\tau) - H(\tau)$$

$$T_p(\tau) = -\tau \ln p - (1-\tau)\ln(1-p)$$

$$H(\tau) = -\tau \ln \tau - (1-\tau)\ln(1-\tau)$$

Letting $\tau = p + \epsilon$ and after expanding the logarithmic terms, we find

$$X = \frac{\epsilon^2}{2\sigma^2} - \frac{\epsilon^3}{6}\left(\frac{1}{p^2} - \frac{1}{q^2}\right) + \frac{\epsilon^4}{12}\left(\frac{1}{p^3} + \frac{1}{q^3}\right)$$

$$+ \ldots + \frac{\epsilon^n}{n(n-1)}\left(\frac{(-1)^n}{p^{n-1}} + \frac{1}{q^{n-1}}\right) + \ldots$$

Note that for sufficiently small ϵ, only the term $\epsilon^2/2\sigma^2$ is significant and the Chernoff and central limit theorem arguments yield the same exponent. In general, the Chernoff exponent (which is correct) is smaller than the central limit theorem exponent for $\epsilon > 0$ and larger for $\epsilon < 0$. In our example, $p = 10^{-2}$, $\epsilon = .05p$, $q = .99$, and, for $\tau > p$,

$$X = \frac{(.05p)^2}{2p(1-p)} - \frac{(.05p)^3}{6p^2} + \cdots$$

$$\approx 1.26 \times 10^{-5} - 2.5 \times 10^{-7} + \cdots$$

The remaining terms may be neglected; indeed only the first term

2.38 continued

is significant. We have

$$e^{-NX} = 10^{-3}$$

$$\underline{\underline{N = 6.12 \times 10^5}}$$

which is a conservative estimate. Note that this result is 2 orders of magnitude smaller than that obtained with the Chebyshev inequality.

2.39

a. Given that a specific message, say m_1, is transmitted, then the $\{r_i\}$ are a set of statistically independent Gaussian random variables, each with mean $\overline{r_i} = -\sqrt{E}$ and variance σ^2. Thus, y is also Gaussian with

$$\overline{y} = \sum_{i=1}^{N} \overline{r_i} = -N\sqrt{E}$$

$$\sigma_y^{\,2} = \sum_{i=1}^{N} \sigma^2(r_i) = N\sigma^2$$

However,

$$P[\mathcal{E}] = P[\mathcal{E}\,|\,m_1] = P[y>0\,|\,m_1] = Q\left(\frac{|\overline{y}|}{\sigma_y}\right) = Q\left(\sqrt{\frac{NE}{\sigma^2}}\right)$$

$$P[\mathcal{E}] < \exp[-NE/2\sigma^2]$$

b. Note that p is just the error probability in (a) when N = 1; hence,

$$p = Q\left(\sqrt{E/\sigma^2}\right)$$

Using Chernoff arguments,

$$P[\mathcal{E}] = P[\mathcal{E}\,|\,m_1] = P[x>0\,|\,m_1] = \overline{u_{-1}(x)} < \overline{e^{\lambda x}} \quad , \lambda > 0$$

in which u_{-1} is the unit step function and the overhead bar indicates an expectation conditioned on m_1. However,

$$\overline{e^{\lambda x}} = \overline{e^{\lambda \sum_{j=1}^{N} x_i}} = \prod_{j=1}^{N} \overline{e^{\lambda x_i}} = \left[(1-p)e^{-\lambda} + pe^{\lambda}\right]^N$$

where

$$\frac{d}{d\lambda}\left[(1-p)e^{-\lambda} + pe^{\lambda}\right] = 0 = -(1-p)e^{-\lambda} + pe^{\lambda}$$

2.39 continued

or

$$e^{\lambda} = \sqrt{\frac{1-p}{p}} \quad \geqslant 1.0$$

Thus,

$$P[\mathcal{E}] < \left[2\sqrt{(1-p)p}\right]^N = e^{N \ln\sqrt{4p(1-p)}}$$

Now,

$$p = \int_{\sqrt{E/\sigma^2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2}\, d\alpha = \frac{1}{2} - \int_{0}^{\sqrt{E/\sigma^2}} \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2}\, d\alpha$$

$$\leq \frac{1}{2} - I$$

in which

$$I = \frac{1}{\sqrt{2\pi}} \int_{0}^{\sqrt{E/\sigma^2}} \left(1 - \frac{\alpha^2}{2}\right) d\alpha$$

$$= \sqrt{\frac{E}{2\pi\sigma^2}} \left(1 - \frac{E}{6\sigma^2}\right)$$

Also,

$$\ln\sqrt{4p(1-p)} \leq \frac{1}{2} \ln(1 - 4I^2) \leq -2I^2, \quad I \geq 0.$$

Hence,

$$P[\mathcal{E}] < e^{-2NI^2}$$

$$= \exp\left\{-N \frac{E}{\pi\sigma^2}\left(1 - \frac{E}{6\sigma^2}\right)^2\right\}$$

Thus, for $\frac{E}{6\sigma^2} << 1$, quantization introduces a degradation of $2/\pi$.