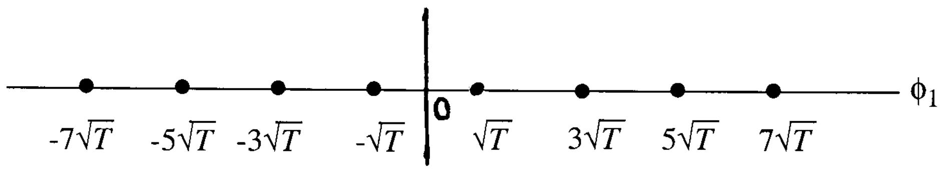
$$= A_i^2 T, \qquad A_i = \pm 1, \pm 3, \pm 5, \pm 7$$

The basis function is given by

$$\phi_1(t) = \frac{s_i(t)}{\sqrt{E_i}} = \frac{s_i(t)}{A_i\sqrt{T}}$$

The signal-space diagram of the 8-level PAM signal is as follows:



Problem 5.3

Consider the signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ shown in Fig. 1a. We wish to use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for this set of signals.

Step 1 We note that the energy of signal $s_1(t)$ is

$$E_1 = \int_0^T s_1^2(t)dt$$

$$= \int_0^{T/3} (1)^2 dt$$

$$= \frac{T}{3}$$

The first basis function $\phi_1(t)$ is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{Bmatrix} \sqrt{3/T}, & 0 \le t \le T/3 \\ 0, & \text{otherwise} \end{Bmatrix}$$

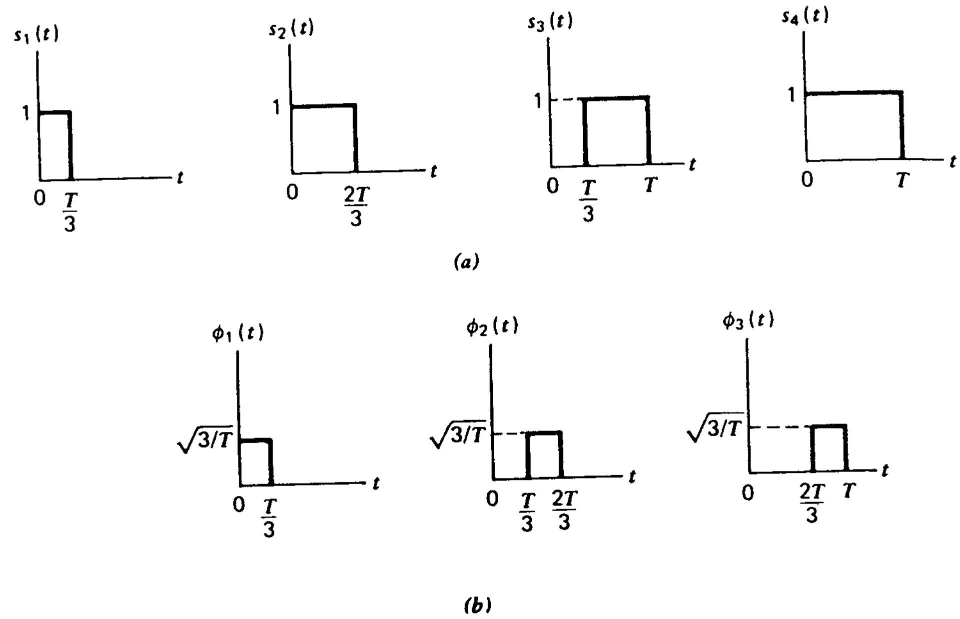


Figure 1

Step 2 Evaluating the projection of $s_2(t)$ onto $\phi_1(t)$, we find that

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

$$= \int_0^{T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The energy of signal $s_2(t)$ is

$$E_2 = \int_0^T s_2^2(t)$$

$$= \int_0^{2T/3} (1)^2 dt$$

$$= \frac{2T}{3}$$

The second basis function $\phi_2(t)$ is therefore

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$= \begin{cases} \sqrt{3/T}, & T/3 \le 2T/3 \\ 0 & \text{otherwise} \end{cases}$$

Step 3 Evaluating the projection of $s_3(t)$ onto $\phi_1(t)$,

$$s_{31} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= 0$$

and the coefficient s_{32} equals

$$s_{32} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= \int_{T/3}^{2T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The corresponding value of the intermediate function $g_i(t)$, with $i = 3$, is therefore

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$= \begin{cases} 1, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

Hence, the third basis function $\phi_3(t)$ is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} \sqrt{3/T}, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

The orthogonalization process is now complete.

The three basis functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ form an orthonormal set, as shown in Fig. 1b. In this example, we thus have $M = 4$ and $N = 3$, which means that the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ described in Fig. 1a do not form a linearly independent set. This is readily confirmed by noting that $s_4(t) = s_1(t) + s_3(t)$. Moreover, we note that any of these four signals can be expressed as a linear combination of the three basis functions, which is the essence of the Gram-Schmidt orthogonalization procedure.

Problem 5.4

(a) We first observe that $s_1(t)$, $s_2(t)$ and $s_3(t)$ are linearly independent.

The energy of $s_1(t)$ is

$$E_1 = \int_0^1 (2)^2 dt = 4$$

The first basis function is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwsie} \end{cases}$$

Define

$$s_{21} = \int_0^T s_2(t)\ \phi_1(t)dt$$

$$= \int_0^1 (-4)(1)dt = -4$$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$= \begin{cases} -4, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the second basis function is

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}$$

$$= \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$s_{31} = \int_0^T s_3(t)\ \phi_1(t)dt$$

$$= \int_0^1 (3)(1)dt = 3$$

$$s_{32} = \int_T^{2T} s_3(t)\ \phi_2(t)dt$$

$$= \int_1^2 (3)(-1)dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\ \phi_1(t) - s_{32}\ \phi_2(t)$$

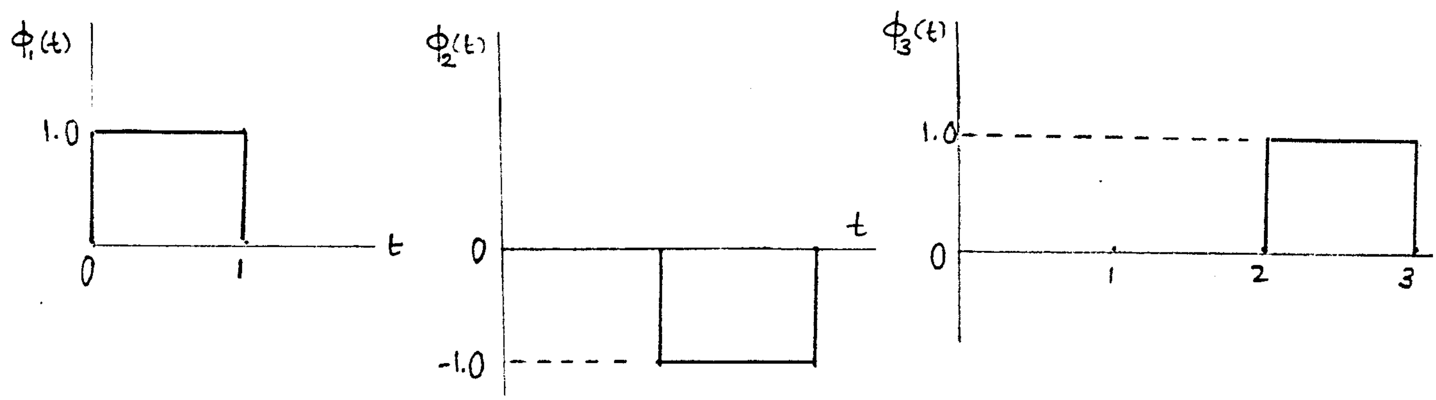
$$= \begin{cases} 3, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the thrid basis function is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

The three basis functions are as follows (graphically)



(b) $s_1(t) = 2\phi_1(t)$

$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$

$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$

Problem 5.5

Signals $s_1(t)$ and $s_2(t)$ are orthogonal to each other. The energy of $s_1(t)$ is

$$E_1 = \int_0^{T/2} 1^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

The energy of $s_2(t)$ is

$$E_2 = \int_0^{T} 1^2 dt = T$$

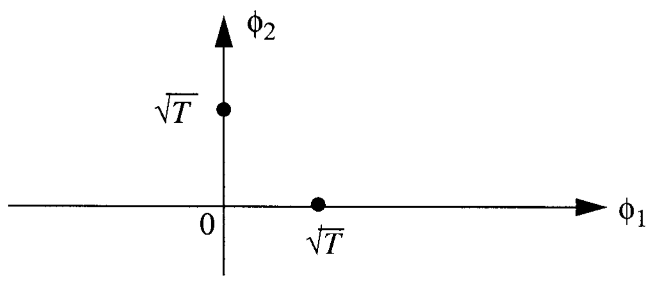
To represent the orthogonal signals $s_1(t)$ and $s_2(t)$, we need two basis functions. The first basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The second basis function is given by

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}$$

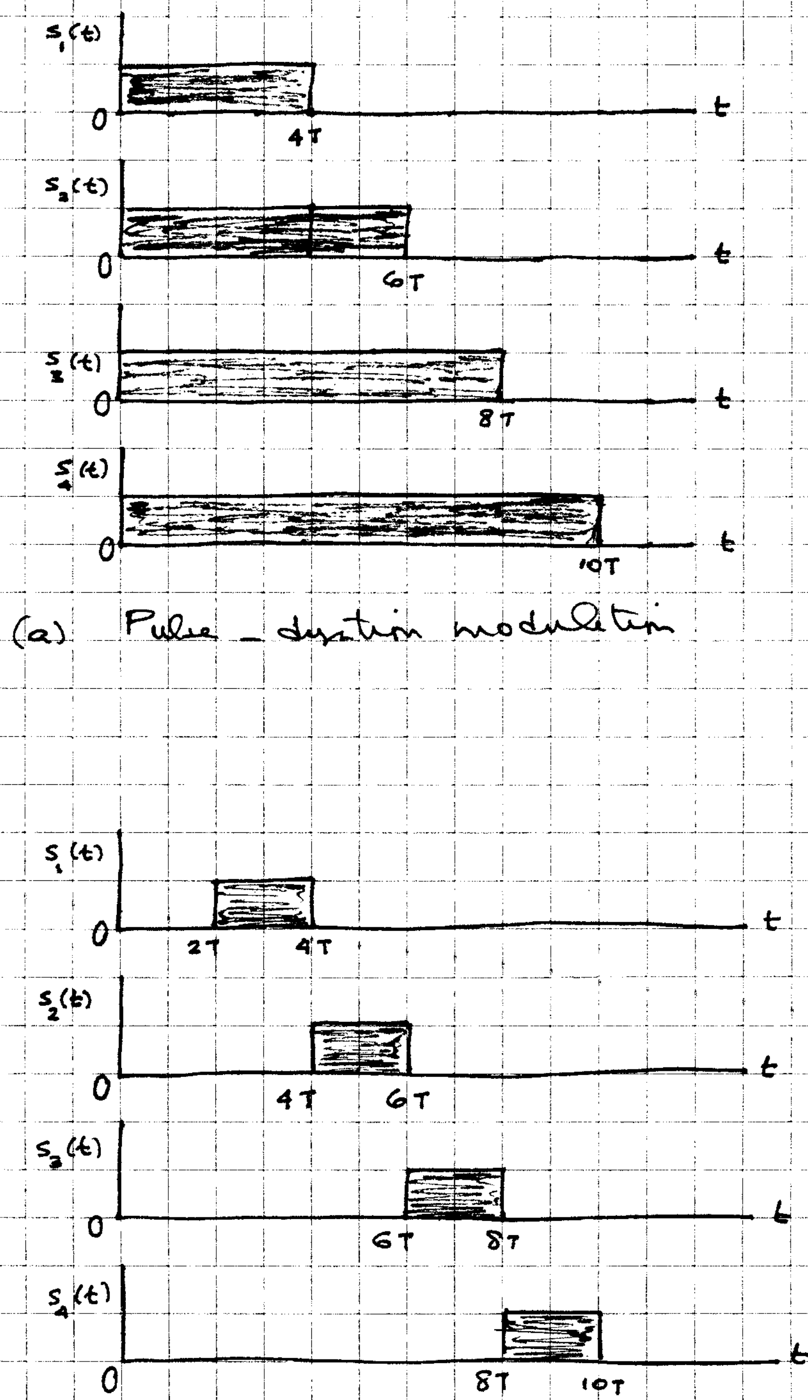
The signal-space diagram for $s_1(t)$ and $s_2(t)$ is as shown below:



Problem 5.6

The common properties of PDM and PPM are as follows: In both cases a time parameter of the pulse is modulated and the pulses have a constant amplitude. In PDM, the samples of the message signals are used to vary the duration of the individual pulses, as illustrated in Fig. 1a for $M = 4$ on the next page. In PPM, the position of the pulse is varied in accordance with the message, while keeping the duration of the pulse constant, as illustrated in Fig. 1b for $M = 4$.

From these two illustrative figures, it is perfectly clear that the set of PDM signals is nonorthogonal, whereas the PDM signals form an orthogonal set.



(b) Pulse-position modulation

Figure 1

Problem 5.7

(a) The biorthogonal signals are defined as the negatives of orthogonal signals. Consider for example the two orthogonal signals $s_1(t)$ and $s_2(t)$ defined as follows:

$$s_1(t) = \sqrt{E}\phi_1(t)$$

$$s_2(t) = \sqrt{E}\phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ are orthonormal basis functions. The biorthogonal signals are given by $-s_1(t)$ and $-s_2(t)$, which are respectively expressed in terms of the basis functions as $\sqrt{E}\phi_1(t)$ and $-\sqrt{E}\phi_2(t)$. Hence, the inclusion of these two biorthogonal signals leaves the dimensionality of the signal-space diagram unchanged. This result holds for the general case of M orthogonal signals.

(b) The signal-space diagram for the biorthogonal signals corresponding to those shown in Fig. P5.5 is as shown in Fig. 1a. Incorporating this diagram with that of the solution to Problem 5.5, we get the 4-signal constellation shown in Fig. 1b.

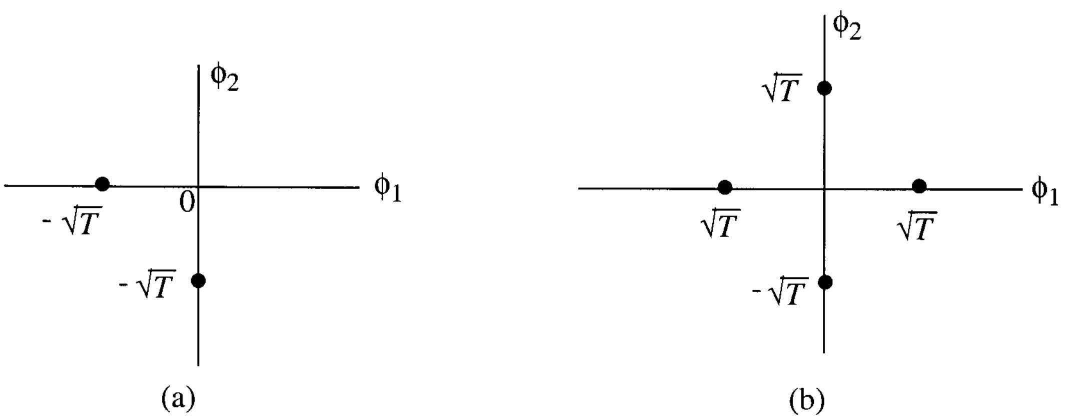


Figure 1

Problem 5.8

(a) A pair of signals $s_i(t)$ and $s_k(t)$, belonging to an N-dimensional signal space, can be represented as linear combinations of N orthonormal basis functions. We thus write

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad \begin{matrix} 0 \le t \le T \\ i = 1, 2 \end{matrix} \tag{1}$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \qquad \begin{matrix} i = 1, 2 \\ j = 1, 2 \end{matrix} \tag{2}$$

The real-valued basis functions $\phi_1(t)$ and $\phi_2(t)$ are orthonormal. Hence,

$$\int_0^T \phi_i(t)\phi_j(t) \quad = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The set of coefficients $\{s_{ij}\}_{j=1}^{N}$ may be viewed as an N-dimensional vector defined by

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \ldots, M \tag{4}$$

where M is the number of signals in the setl, with $M \geq N$. The inner product of the pair of signal $s_i(t)$ and $s_k(t)$ is given by

$$\int_0^T s_i(t)s_k(t)dt \tag{5}$$

By substituting (1) in (5), we get the following result for the inner product:

$$\int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right] \left[\sum_{l=1}^{N} s_{kl}\phi_l(t)\right] dt$$

$$= \sum_{j=1}^{N} \sum_{l=1}^{N} s_{ij}s_{kl}\int_0^T \phi_j(t)\phi_l(t)dt \tag{6}$$

Since the $\phi_j(t)$ form an orthonormal set, then, in accordance with the two conditions of Eq. (3) and (4), the inner product of $s_i(t)$ and $s_k(t)$ reduces to

$$\int_0^T s_i(t)s_k(t)dt = \sum_{j=1}^{N} s_{ij}s_{kj}$$

$$= \mathbf{s}_i^T \mathbf{s}_k$$

(b) Consider next the squared Euclidean distance between $\mathbf{s}_i$ and $\mathbf{s}_k$, which can be expressed as follows:

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = (\mathbf{s}_i - \mathbf{s}_k)^T (\mathbf{s}_i - \mathbf{s}_k)$$

$$= \mathbf{s}_i^T \mathbf{s}_i + \mathbf{s}_k^T \mathbf{s}_k - 2\mathbf{s}_i^T \mathbf{s}_k$$

$$= \int_0^T s_i^2(t)dt + \int_0^T s_k^2(t)dt - 2\int_0^T s_i(t)s_k dt$$

$$= \int_0^T (s_i(t) - s_k(t))^2 dt$$

Problem 5.9

Consider the pair of complex-valued signals $s_1(t)$ and $s_2(t)$, which are defined by

$$s_1(t) = a_{11}\phi_1(t) + a_{12}\phi_2(t) \qquad (1)$$

$$s_2(t) = a_{21}\phi_1(t) + a_{22}\phi_2(t) \qquad (2)$$

The basis functions $\phi_1(t)$ and $\phi_2(t)$ are real-valued and the coefficients a_{11}, a_{12}, a_{21} and a_{22} are complex-valued. We may denote the complex-valued coefficients as follows:

$a_{11} = \alpha_{11} + j\beta_{11}$
$a_{12} = \alpha_{12} + j\beta_{12}$
$a_{21} = \alpha_{21} + j\beta_{21}$
$a_{22} = \alpha_{22} + j\beta_{22}$

On this basis, we may represent the signals $s_1(t)$ and $s_2(t)$ by the following respective pair of vectors: